

Modeling and Verification of Probabilistic Systems

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June 23, 2014

Negative exponential distribution

Density of exponential distribution

The density of an *exponentially distributed* r.v. Y with *rate* $\lambda \in \mathbb{R}_{>0}$ is:

$$f_Y(x) = \lambda \cdot e^{-\lambda \cdot x} \quad \text{for } x > 0 \quad \text{and } f_Y(x) = 0 \text{ otherwise}$$

The cumulative distribution of r.v. Y with rate $\lambda \in \mathbb{R}_{>0}$ is:

$$F_Y(d) = \int_0^d \lambda \cdot e^{-\lambda \cdot x} dx = [-e^{-\lambda \cdot x}]_0^d = 1 - e^{-\lambda \cdot d}.$$

The rate $\lambda \in \mathbb{R}_{>0}$ uniquely determines an exponential distribution.

Variance and expectation

Let r.v. Y be exponentially distributed with rate $\lambda \in \mathbb{R}_{>0}$. Then:

- ▶ Expectation $E[Y] = \int_0^\infty x \cdot \lambda \cdot e^{-\lambda \cdot x} dx = \frac{1}{\lambda}$
- ▶ Variance $\text{Var}[Y] = \int_0^\infty (x - E[X])^2 \lambda \cdot e^{-\lambda \cdot x} dx = \frac{1}{\lambda^2}$

Overview

- 1 Recall: continuous-time Markov chains
- 2 Transient distribution
- 3 Uniformization
- 4 Strong and weak bisimulation
- 5 Computing transient probabilities
- 6 Summary

Continuous-time Markov chain

Continuous-time Markov chain

A **CTMC** is a tuple $(S, \mathbf{P}, r, \ell_{\text{init}}, AP, L)$ where

- ▶ $(S, \mathbf{P}, \ell_{\text{init}}, AP, L)$ is a DTMC, and
- ▶ $r : S \rightarrow \mathbb{R}_{>0}$, the **exit-rate function**

Let $\mathbf{R}(s, s') = \mathbf{P}(s, s') \cdot r(s)$ be the transition rate of transition (s, s')

Interpretation

- ▶ **residence** time in state s is exponentially distributed with **rate** $r(s)$.
- ▶ phrased alternatively, the **average** residence time of state s is $\frac{1}{r(s)}$.

CTMC semantics

Enabledness

The probability that transition $s \rightarrow s'$ is *enabled* in $[0, t]$ is $1 - e^{-R(s,s') \cdot t}$.

State-to-state timed transition probability

The probability to *move* from non-absorbing s to s' in $[0, t]$ is:

$$\frac{R(s, s')}{r(s)} \cdot (1 - e^{-r(s) \cdot t}).$$

Residence time distribution

The probability to *take some* outgoing transition from s in $[0, t]$ is:

$$\int_0^t r(s) \cdot e^{-r(s) \cdot x} dx = 1 - e^{-r(s) \cdot t}$$

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Transient distribution of a CTMC

Transient state probability

Let $X(t)$ denote the state of a CTMC at time $t \in \mathbb{R}_{\geq 0}$. The probability to be in state s at time t is defined by:

$$\begin{aligned} p_s(t) &= Pr\{X(t) = s\} \\ &= \sum_{s' \in S} Pr\{X(0) = s'\} \cdot Pr\{X(t) = s \mid X(0) = s'\} \end{aligned}$$

Theorem: transient distribution as linear differential equation

The *transient* probability vector $\underline{p}(t) = (p_{s_1}(t), \dots, p_{s_k}(t))$ satisfies:

$$\underline{p}'(t) = \underline{p}(t) \cdot (\mathbf{R} - \mathbf{r}) \quad \text{given} \quad \underline{p}(0)$$

where \mathbf{r} is the diagonal matrix of vector \underline{r} .

Transient distribution theorem

Theorem: transient distribution as linear differential equation

The *transient* probability vector $\underline{p}(t) = (p_{s_1}(t), \dots, p_{s_k}(t))$ satisfies:

$$\underline{p}'(t) = \underline{p}(t) \cdot (\mathbf{R} - \mathbf{r}) \quad \text{given} \quad \underline{p}(0)$$

where \mathbf{r} is the diagonal matrix of vector \underline{r} .

Proof:

On the blackboard.

Computing transient probabilities

The transient probability vector $\underline{p}(t) = (p_{s_1}(t), \dots, p_{s_k}(t))$ satisfies:

$$\underline{p}'(t) = \underline{p}(t) \cdot (\mathbf{R}-\mathbf{r}) \quad \text{given} \quad \underline{p}(0).$$

Solution using standard knowledge yields: $\underline{p}(t) = \underline{p}(0) \cdot e^{(\mathbf{R}-\mathbf{r}) \cdot t}$.

Computing a matrix exponential

First attempt: use [Taylor-Maclaurin](#) expansion. This yields

$$\underline{p}(t) = \underline{p}(0) \cdot e^{(\mathbf{R}-\mathbf{r}) \cdot t} = \underline{p}(0) \cdot \sum_{i=0}^{\infty} \frac{((\mathbf{R}-\mathbf{r}) \cdot t)^i}{i!}$$

But: [numerical instability](#) due to fill-in of $(\mathbf{R}-\mathbf{r})^i$ in presence of positive and negative entries in the matrix $\mathbf{R}-\mathbf{r}$.

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- 1 Recall: continuous-time Markov chains
- 2 Transient distribution
- 3 **Uniformization**
- 4 Strong and weak bisimulation
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Uniformization

Let CTMC $\mathcal{C} = (S, \mathbf{P}, r, \iota_{\text{init}}, AP, L)$ with S finite.

Uniform CTMC

CTMC \mathcal{C} is [uniform](#) if $r(s) = r$ for all $s \in S$ for some $r \in \mathbb{R}_{>0}$.

Uniformization

[Gross and Miller, 1984]

Let $r \in \mathbb{R}_{>0}$ such that $r \geq \max_{s \in S} r(s)$. Then $\text{unif}(r, \mathcal{C})$ is the tuple $(S, \bar{\mathbf{P}}, \bar{r}, \iota_{\text{init}}, AP, L)$ with $\bar{r}(s) = r$ for all $s \in S$, and:

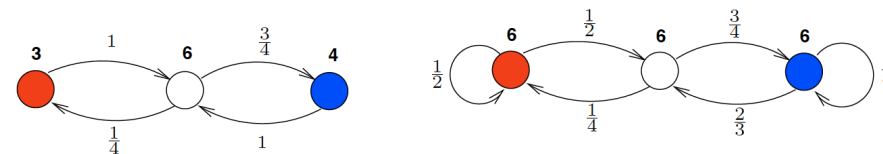
$$\bar{\mathbf{P}}(s, s') = \frac{r(s)}{r} \cdot \mathbf{P}(s, s') \text{ if } s' \neq s \quad \text{and} \quad \bar{\mathbf{P}}(s, s) = \frac{r(s)}{r} \cdot \mathbf{P}(s, s) + 1 - \frac{r(s)}{r}.$$

Uniformization: example

Uniformization

Let $r \in \mathbb{R}_{>0}$ such that $r \geq \max_{s \in S} r(s)$. Then $\text{unif}(r, \mathcal{C}) = (S, \bar{\mathbf{P}}, \bar{r}, \iota_{\text{init}}, AP, L)$ with $\bar{r}(s) = r$ for all $s \in S$, and:

$$\bar{\mathbf{P}}(s, s') = \frac{r(s)}{r} \cdot \mathbf{P}(s, s') \text{ if } s' \neq s \quad \text{and} \quad \bar{\mathbf{P}}(s, s) = \frac{r(s)}{r} \cdot \mathbf{P}(s, s) + 1 - \frac{r(s)}{r}.$$



CTMC \mathcal{C} and its uniformized counterpart $\text{unif}(6, \mathcal{C})$

Uniformization: intuition

Uniformization

Let $r \in \mathbb{R}_{>0}$ such that $r \geq \max_{s \in S} r(s)$. Then $\text{unif}(r, \mathcal{C}) = (S, \bar{\mathbf{P}}, \bar{r}, \iota_{\text{init}}, AP, L)$ with $\bar{r}(s) = r$ for all $s \in S$, and:

$$\bar{\mathbf{P}}(s, s') = \frac{r(s)}{r} \cdot \mathbf{P}(s, s') \text{ if } s' \neq s \quad \text{and} \quad \bar{\mathbf{P}}(s, s) = \frac{r(s)}{r} \cdot \mathbf{P}(s, s) + 1 - \frac{r(s)}{r}.$$

Intuition

- ▶ Fix all exit rates to (at least) the **maximal** exit rate r occurring in CTMC \mathcal{C} .
- ▶ Thus, $\frac{1}{r}$ is the **shortest** mean residence time in the CTMC \mathcal{C} .
- ▶ Then **normalize** the residence time of all states with respect to r as follows:
 1. replace an average residence time $\frac{1}{r(s)}$ by a shorter (or equal) one, $\frac{1}{r}$
 2. decrease the transition probabilities by a factor $\frac{r(s)}{r}$, and
 3. increase the self-loop probability by a factor $\frac{r-r(s)}{r}$

That is, **slow down** state s whenever $r(s) < r$.

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Strong bisimulation on DTMCs

Probabilistic bisimulation

[Larsen & Skou, 1989]

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an **equivalence**. Then: R is a **probabilistic bisimulation** on S if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $\mathbf{P}(s, C) = \mathbf{P}(t, C)$ for all equivalence classes $C \in S/R$

where $\mathbf{P}(s, C) = \sum_{s' \in C} \mathbf{P}(s, s')$.

For states in R , the probability of moving by a single transition to some equivalence class is equal.

Probabilistic bisimilarity

Let \mathcal{D} be a DTMC and s, t states in \mathcal{D} . Then: s is **probabilistically bisimilar** to t , denoted $s \sim_p t$, if there **exists** a probabilistic bisimulation R with $(s, t) \in R$.

Strong bisimulation on CTMCs

Probabilistic bisimulation

[Buchholz, 1994]

Let $\mathcal{C} = (S, \mathbf{P}, r, \iota_{\text{init}}, AP, L)$ be a CTMC and $R \subseteq S \times S$ an **equivalence**. Then: R is a **probabilistic bisimulation** on S if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $r(s) = r(t)$, and
3. $\mathbf{P}(s, C) = \mathbf{P}(t, C)$ for all equivalence classes $C \in S/R$

The last two conditions amount to $\mathbf{R}(s, C) = \mathbf{R}(t, C)$ for all equivalence classes $C \in S/R$.

Probabilistic bisimilarity

Let \mathcal{C} be a CTMC and s, t states in \mathcal{C} . Then: s is **probabilistically bisimilar** to t , denoted $s \sim_m t$, if there **exists** a probabilistic bisimulation R with $(s, t) \in R$.

Weak bisimulation on DTMCs

Weak probabilistic bisimulation

[Baier & Hermanns, 1996]

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an **equivalence**. Then: R is a **weak probabilistic bisimulation** on S if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. if $\mathbf{P}(s, [s]_R) < 1$ and $\mathbf{P}(t, [t]_R) < 1$, then:

$$\frac{\mathbf{P}(s, C)}{1 - \mathbf{P}(s, [s]_R)} = \frac{\mathbf{P}(t, C)}{1 - \mathbf{P}(t, [t]_R)} \quad \text{for all } C \in S/R, C \neq [s]_R = [t]_R.$$

3. s can reach a state outside $[s]_R$ iff t can reach a state outside $[t]_R$.

For states in R , the **conditional** probability of moving by a single transition to **another** equivalence class is equal. In addition, either all states in an equivalence class C almost surely stay there, or have an option to escape from C .

Weak bisimulation on DTMCs

Weak probabilistic bisimulation

[Baier & Hermanns, 1996]

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an **equivalence**. Then: R is a **weak probabilistic bisimulation** on S if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. if $\mathbf{P}(s, [s]_R) < 1$ and $\mathbf{P}(t, [t]_R) < 1$, then:

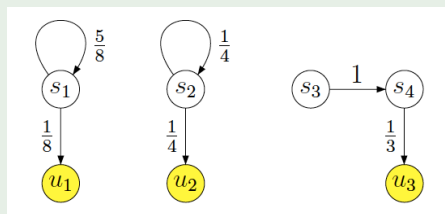
$$\frac{\mathbf{P}(s, C)}{1 - \mathbf{P}(s, [s]_R)} = \frac{\mathbf{P}(t, C)}{1 - \mathbf{P}(t, [t]_R)} \quad \text{for all } C \in S/R, C \neq [s]_R = [t]_R.$$

3. s can reach a state outside $[s]_R$ iff t can reach a state outside $[t]_R$.

Probabilistic weak bisimilarity

Let \mathcal{D} be a DTMC and s, t states in \mathcal{D} . Then: s is **probabilistically weak bisimilar** to t , denoted $s \approx_p t$, if there **exists** a probabilistic weak bisimulation R with $(s, t) \in R$.

Weak bisimulation on DTMC: example



The equivalence relation R with $S/R = \{ \{s_1, s_2, s_3, s_4\}, \{u_1, u_2, u_3\} \}$ is a weak bisimulation. This can be seen as follows. For $C = \{u_1, u_2, u_3\}$ and s_1, s_2, s_4 with $\mathbf{P}(s_i, [s_i]_R) < 1$ we have:

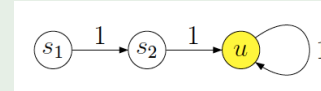
$$\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, [s_1]_R)} = \frac{1/8}{1 - 5/8} = \frac{1/4}{1 - 1/4} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, [s_2]_R)} = \frac{1/3}{1} = \frac{\mathbf{P}(s_4, C)}{1 - \mathbf{P}(s_4, [s_4]_R)}.$$

Note that $\mathbf{P}(s_3, [s_3]_R) = 1$. Since s_3 can reach a state outside $[s_3]$ as s_1, s_2 and s_4 , it follows that $s_1 \approx_p s_2 \approx_p s_3 \approx_p s_4$.

Reachability condition

Remark

Consider the following DTMC:



It is not difficult to establish $s_1 \approx s_2$. Note: $\mathbf{P}(s_1, [s_1]) = 1$, but $\mathbf{P}(s_2, [s_2]_R) < 1$. Both s_1 and s_2 can reach a state outside $[s_1]_R = [s_2]_R$. The reachability condition is essential to establish $s_1 \approx s_2$ and cannot be dropped: otherwise s_1 and s_2 would be weakly bisimilar to an equally labelled absorbing state.

Weak bisimulation on CTMCs

Weak probabilistic bisimulation

[Bravetti, 2002]

Let $\mathcal{C} = (S, \mathbf{P}, r, \iota_{\text{init}}, AP, L)$ be a CTMC and $R \subseteq S \times S$ an **equivalence**.

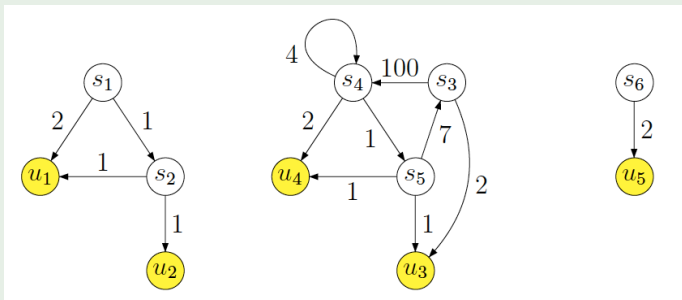
Then: R is a **weak probabilistic bisimulation** on S if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $\mathbf{R}(s, C) = \mathbf{R}(t, C)$ for all $C \in S/R$ with $C \neq [s]_R = [t]_R$

Weak probabilistic bisimilarity

Let \mathcal{C} be a CTMC and s, t states in \mathcal{C} . Then: s is **weak probabilistically bisimilar** to t , denoted $s \approx_m t$, if there **exists** a weak probabilistic bisimulation R with $(s, t) \in R$.

Weak bisimulation on CTMCs: example



Equivalence relation R with $S/R = \{ \{s_1, s_2, s_3, s_4, s_5, s_6\}, \{u_1, u_2, u_3, u_4, u_5\} \}$ is a weak bisimulation on the CTMC depicted above. This can be seen as follows. For $C = \{u_1, u_2, u_3, u_4, u_5\}$, we have that all s -states enter C with rate 2. The rates between the s -states are not relevant.

A useful lemma

Let \mathcal{C} be a CTMC and R an equivalence relation on S with $(s, t) \in R$. Then: the following two statements are equivalent:

1. If $\mathbf{P}(s, [s]_R) < 1$ and $\mathbf{P}(t, [t]_R) < 1$ then for all $C \in S/R$, $C \neq [s]_R = [t]_R$:

$$\frac{\mathbf{P}(s, C)}{1 - \mathbf{P}(s, [s]_R)} = \frac{\mathbf{P}(t, C)}{1 - \mathbf{P}(t, [t]_R)} \quad \text{and} \quad \mathbf{R}(s, S \setminus [s]_R) = \mathbf{R}(t, S \setminus [t]_R)$$

2. $\mathbf{R}(s, C) = \mathbf{R}(t, C)$ for all $C \in S/R$ with $C \neq [s]_R = [t]_R$.

Proof:

Left as an exercise.

Properties (without proof)

Strong and weak bisimulation in uniform CTMCs

For all uniform CTMCs \mathcal{C} and states s, u in \mathcal{C} , we have:

$$s \sim_m u \quad \text{iff} \quad s \approx_m u \quad \text{iff} \quad s \sim_p u.$$

For any CTMC \mathcal{C} , we have: $\mathcal{C} \approx_m \text{unif}(r, \mathcal{C})$ with $r \geq \max_{s \in S} r(s)$.

Preservation of transient probabilities

For all CTMCs \mathcal{C} with states s, u in \mathcal{C} and $t \in \mathbb{R}_{\geq 0}$, we have:

$$s \approx_m u \quad \text{implies} \quad \underline{p}(t) = \underline{p}(t)$$

where $\underline{p}(0) = \mathbf{1}_s$ and $\underline{p}(0) = \mathbf{1}_u$ where $\mathbf{1}_s$ is the characteristic function for state s , i.e., $\mathbf{1}_s(s') = 1$ iff $s = s'$.

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Computing transient probabilities

The transient probability vector $\underline{p}(t) = (p_{s_1}(t), \dots, p_{s_k}(t))$ satisfies:

$$\underline{p}'(t) = \underline{p}(t) \cdot (\mathbf{R} - \mathbf{r}) \quad \text{given} \quad \underline{p}(0).$$

Standard knowledge yields: $\underline{p}(t) = \underline{p}(0) \cdot e^{(\mathbf{R} - \mathbf{r}) \cdot t}$.

As uniformization preserves transient probabilities, we replace $\mathbf{R} - \mathbf{r}$ by its variant for the uniformized CTMC, i.e., $\bar{\mathbf{R}} - \bar{\mathbf{r}}$. We have:

$$\bar{\mathbf{R}}(s, s') = \bar{\mathbf{P}}(s, s') \cdot \bar{\mathbf{r}}(s) = \bar{\mathbf{P}}(s, s') \cdot r \quad \text{and} \quad \bar{\mathbf{r}} = \mathbf{I} \cdot r.$$

Thus:

$$\underline{p}(0) \cdot e^{(\bar{\mathbf{R}} - \bar{\mathbf{r}}) \cdot t} = \underline{p}(0) \cdot e^{(\bar{\mathbf{P}} \cdot r - \mathbf{I} \cdot r) \cdot t} = \underline{p}(0) \cdot e^{(\bar{\mathbf{P}} - \mathbf{I}) \cdot r \cdot t} = \underline{p}(0) \cdot e^{-rt} \cdot e^{r \cdot t \cdot \bar{\mathbf{P}}}.$$

Computing transient probabilities

$$\underline{p}(t) = \underline{p}(0) \cdot e^{(\bar{\mathbf{R}} - \bar{\mathbf{r}}) \cdot t} = \underline{p}(0) \cdot e^{(\bar{\mathbf{P}} \cdot r - \mathbf{I} \cdot r) \cdot t} = \underline{p}(0) \cdot e^{(\bar{\mathbf{P}} - \mathbf{I}) \cdot r \cdot t} = \underline{p}(0) \cdot e^{-rt} \cdot e^{r \cdot t \cdot \bar{\mathbf{P}}}.$$

Computing a matrix exponential

Exploit [Taylor-Maclaurin](#) expansion. This yields:

$$\underline{p}(0) \cdot e^{-rt} \cdot e^{r \cdot t \cdot \bar{\mathbf{P}}} = \underline{p}(0) \cdot e^{-rt} \cdot \sum_{i=0}^{\infty} \frac{(r \cdot t)^i}{i!} \cdot \bar{\mathbf{P}}^i = \underline{p}(0) \cdot \sum_{i=0}^{\infty} \underbrace{e^{-r \cdot t} \frac{(r \cdot t)^i}{i!}}_{\text{Poisson prob.}} \cdot \bar{\mathbf{P}}^i$$

As $\bar{\mathbf{P}}$ is a stochastic matrix, computing the matrix exponential $\bar{\mathbf{P}}^i$ is numerically stable.

Intermezzo: Poisson distribution

Poisson distribution

The [Poisson distribution](#) is a discrete probability distribution that expresses the probability of a given number i of events occurring in a fixed interval of time $[0, t]$ if these events occur with a known average rate r and independently of the time since the last event. Formally, the pdf is:

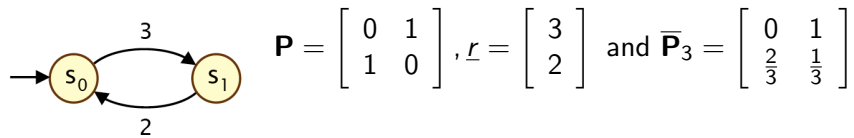
$$f(i; r \cdot t) = e^{-r \cdot t} \frac{(r \cdot t)^i}{i!}$$

where r is the mean of the Poisson distribution.

Remark

The Poisson distribution can be derived as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed.

Transient probabilities: example



$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{r} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \bar{\mathbf{P}}_3 = \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Let initial distribution $\underline{p}(0) = (1, 0)$, and time bound $t=1$. Then:

$$\begin{aligned} \underline{p}(1) &= \underline{p}(0) \cdot \sum_{i=0}^{\infty} e^{-3} \frac{3^i}{i!} \cdot \bar{\mathbf{P}}^i \\ &= (1, 0) \cdot e^{-3} \frac{1}{0!} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (1, 0) \cdot e^{-3} \frac{3}{1!} \cdot \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ &\quad + (1, 0) \cdot e^{-3} \frac{9}{2!} \cdot \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}^2 + \dots \\ &\approx (0.404043, 0.595957) \end{aligned}$$

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Truncating the infinite sum

Computing transient probabilities

$$\underline{p}(t) = \underline{p}(0) \cdot \sum_{i=0}^{\infty} e^{-r \cdot t} \frac{(r \cdot t)^i}{i!} \cdot \bar{\mathbf{P}}^i$$

- ▶ Summation can be truncated *a priori* for a given error bound $\varepsilon > 0$.
- ▶ The *error* that is introduced by truncating at summand k_ε is:

$$\left\| \sum_{i=0}^{\infty} e^{-rt} \frac{(rt)^i}{i!} \cdot \underline{p}(i) - \sum_{i=0}^{k_\varepsilon} e^{-rt} \frac{(rt)^i}{i!} \cdot \underline{p}(i) \right\| = \left\| \sum_{i=k_\varepsilon+1}^{\infty} e^{-rt} \frac{(rt)^i}{i!} \cdot \underline{p}(i) \right\|$$

- ▶ Strategy: choose k_ε minimal such that:

$$\sum_{i=k_\varepsilon+1}^{\infty} e^{-rt} \frac{(rt)^i}{i!} = \sum_{i=0}^{\infty} e^{-rt} \frac{(rt)^i}{i!} - \sum_{i=0}^{k_\varepsilon} e^{-rt} \frac{(rt)^i}{i!} = 1 - \sum_{i=0}^{k_\varepsilon} e^{-rt} \frac{(rt)^i}{i!} \leq \varepsilon$$

Summary

Main points

- ▶ Bisimilar states are equally labelled and their cumulative rate to any equivalence class coincides.
- ▶ Weak bisimilar states have equal conditional probabilities to move to some equivalence class, and can either both leave their class or both can't.
- ▶ Uniformization normalizes the exit rates of all states in a CTMC.
- ▶ Uniformization transforms a CTMC into a weak bisimilar one.
- ▶ Transient distributions are obtained by solving a system of linear differential equations.
- ▶ These equations can be solved conveniently on the uniformized CTMC.