Lower Bounds for Possibly Divergent Probabilistic Programs

Mingshuai Chen

—Joint work with S. Feng, B. L. Kaminski, J.-P. Katoen, H. Su, and N. Zhan—

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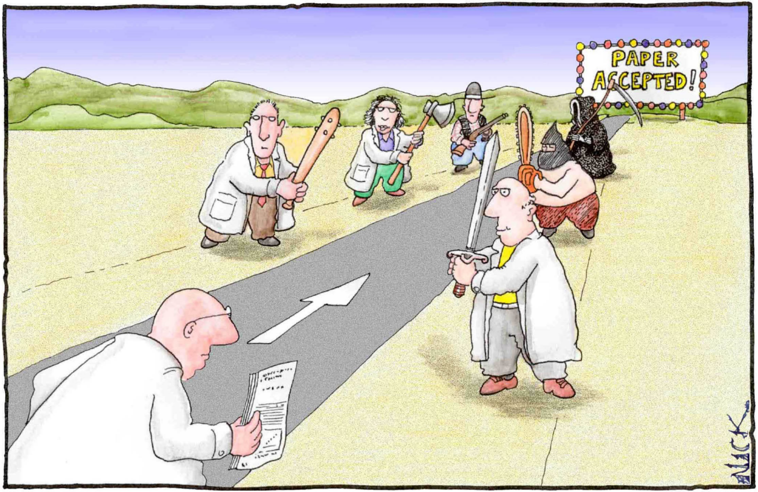
Work in Progress …



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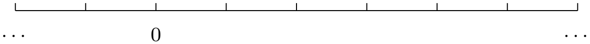

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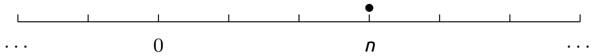

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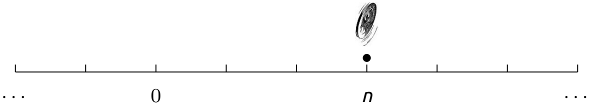

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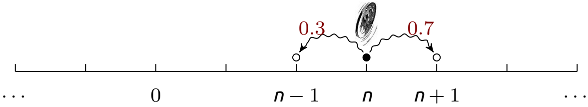

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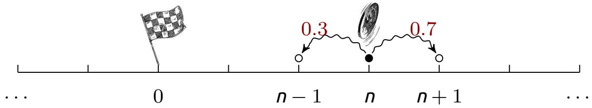

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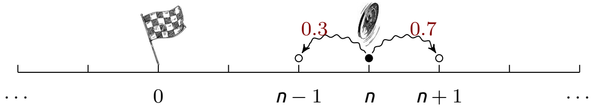
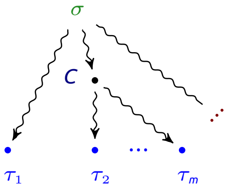


"The crux of probabilistic programming is to treat normal-looking programs as if they were probability distributions."

— Michael Hicks, The PL Enthusiast

Quantitative Reasoning about Probabilistic Loops

[Kozen; McIver, Morgan; Kaminski]



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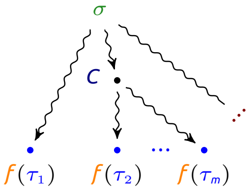

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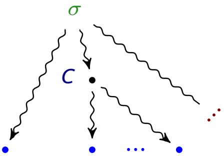

$$\mathsf{wp}[\![C]\!](f)(\sigma) \;\triangleq\; \mathsf{Exp}\Big[\, f(\tau_1) \qquad f(\tau_2) \qquad f(\tau_m) \Big]$$

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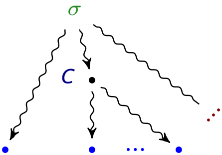

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$$\mathsf{wp}[\![\texttt{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(f) \;=\; \mathsf{lfp}\;\Phi_f \;=\; ?$$

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$$l \;\preceq\; \mathsf{lfp}\,\Phi_f \;\preceq\; u$$

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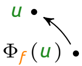

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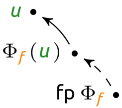

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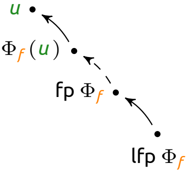

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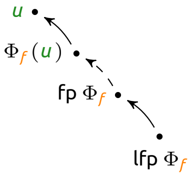

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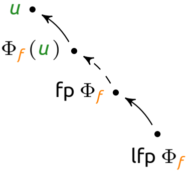

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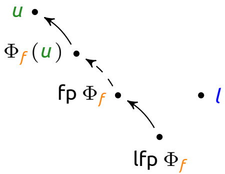

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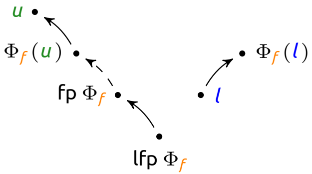

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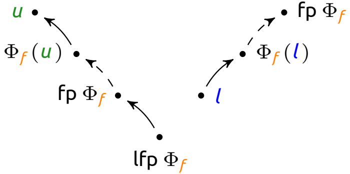

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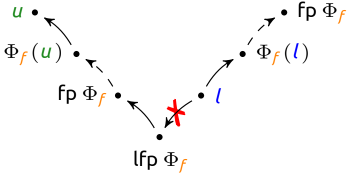

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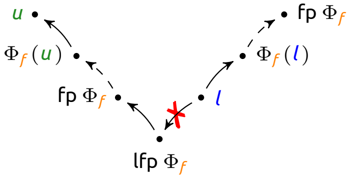

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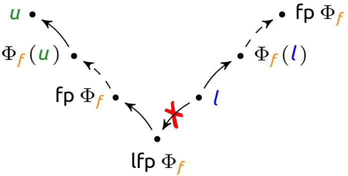
almost-sure termination
bounded expectations
...



A New Proof Rule for Lower Bounds

$$\textit{Loop}\colon\quad \texttt{while}\,(\,\varphi\,)\,\{\,C\,\} \qquad \rightsquigarrow \qquad \textit{Loop}'\colon\quad \texttt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

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- Applicable to *possibly divergent Loop*.

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- Applicable to *possibly divergent Loop*.
- l can be *arbitrarily tight*.
- Reducible to *probabilistic BMC*.
- Easier to ensure *uni. int.* for Loop'.
- …

Example : Biased Random Walk

```
$C_{\mathrm{brw}}$:  while $(0 < n)$ {
                         $n := n - 1\ [0.3]\ n := n + 1$
                     }
```

```
$C_{\mathrm{brw}}^{M}$:  while $(0 < n < M)$ {
                             $n := n - 1\ [0.3]\ n := n + 1$
                         }
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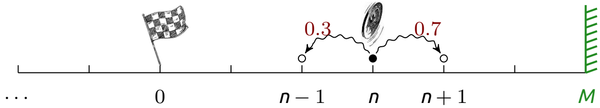

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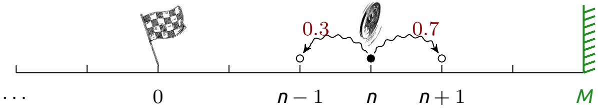

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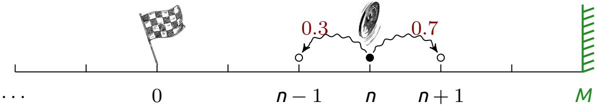

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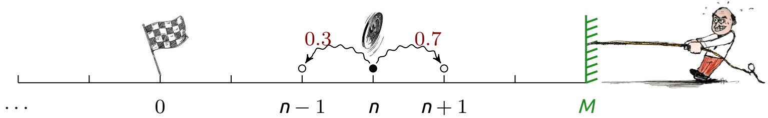

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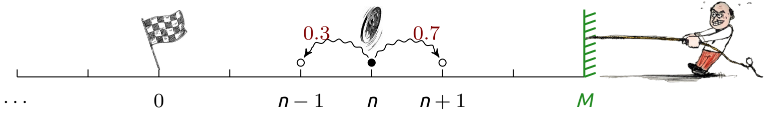

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