Probabilistic Programming

Lecture #7: Fixed Point Theory

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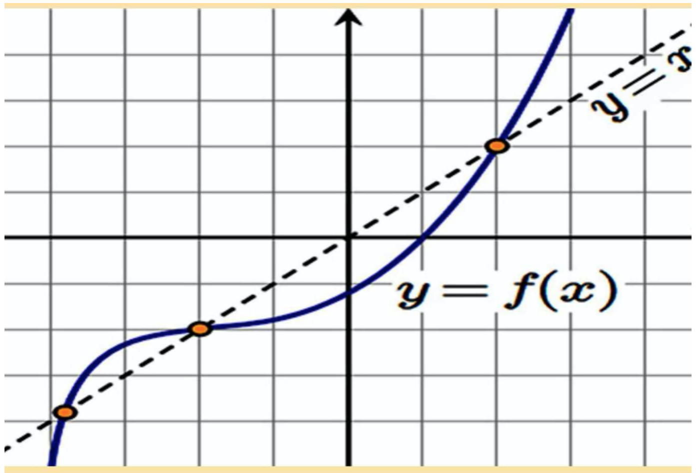
RWTH Lecture Series on Probabilistic Programming 2022-23

Overview

Fixed points



Aims and sufficient conditions

- In denotational program semantics, the semantics of a loop is defined as some fixed point of a mathematical function
  - We will consider this for `pGCL`, but first without probabilities
- Goals:
  - Prove existence of such fixed points
  - Show how they can be "computed" (more exactly: approximated)
- Sufficient conditions:
  - on function domains: complete lattices
  - on functions: monotonicity and Scott continuity

Overview

1. Complete lattices
2. Monotonic and continuous functions
3. Fixpoint theorems

Partial orders

Partial order

A partial order (PO) $(D, \sqsubseteq)$ consists of a set D, called domain, and of a relation $\sqsubseteq \subseteq D \times D$ such that, for every $d_1, d_2, d_3 \in D$:

- $d_1 \sqsubseteq d_1$ (reflexivity)
- $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_3 \implies d_1 \sqsubseteq d_3$ (transitivity)
- $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1 \implies d_1 = d_2$ (antisymmetry)

A PO is called total if, in addition, always $d_1 \sqsubseteq d_2$ or $d_2 \sqsubseteq d_1$.

Examples

1. $(\mathbb{N}, \leqslant)$ is a total partial order
2. $(2^{\mathbb{N}}, \subseteq)$ is a (non-total) partial order
3. $(\mathbb{N}, <)$ is not a partial order (since not reflexive)

Upper and lower bounds

Upper bound

An element $d \in D$ is called an upper bound of $S \subseteq D$, denoted $S \sqsubseteq d$, if $s \sqsubseteq d$ for every $s \in S$.

An upper bound d of $S \subseteq D$ is called least upper bound (LUB) or supremum of S, denoted $d = \bigsqcup S$, if $d \sqsubseteq d'$ for every upper bound d' of S.

Lower bound

An element $d \in D$ is called a lower bound of $S \subseteq D$, denoted $d \sqsubseteq S$, if $d \sqsubseteq s$ for every $s \in S$.

A lower bound d of $S \subseteq D$ is called greatest lower bound (GLB) or infimum of S, denoted $d = \bigsqcap S$, if $d' \sqsubseteq d$ for every lower bound d' of S.

Chains

Chains

$S \subseteq D$ is called a chain in D if, for every $s_1, s_2 \in S$,

$$s_1 \sqsubseteq s_2 \text{ or } s_2 \sqsubseteq s_1.$$

That is, S is a totally ordered subset of D.

A chain $S = s_1 \sqsubseteq s_2 \sqsubseteq s_3 \sqsubseteq \ldots$ is a ascending.
A chain $S = s_1 \sqsupseteq s_2 \sqsupseteq s_3 \sqsupseteq \ldots$ is a descending.

Examples: chains

1. Every subset $S \subseteq \mathbb{N}$ is a chain in $(\mathbb{N}, \leqslant)$.
   It has a LUB (its greatest element) iff it is finite.
2. $\{\varnothing, \{0\}, \{0,1\}, \ldots\}$ is a chain in $(2^{\mathbb{N}}, \subseteq)$ with LUB $\mathbb{N}$.

Complete lattices

Complete lattice

A PO $(D, \sqsubseteq)$ is called a complete lattice, if every subset $S \subseteq D$ has a supremum in D, i.e., $\bigsqcup S \in D$, and an infimum in D, i.e., $\bigsqcap S \in D$.

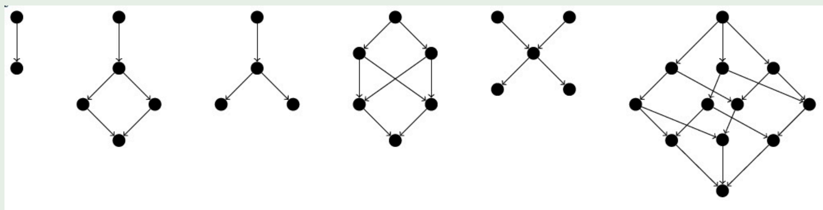
Every complete lattice $(D, \sqsubseteq)$ has a least element $\bot$ and, dually, a greatest element $\top$ which satisfy:

$$\forall d \in D. \quad \bot \sqsubseteq d \sqsubseteq \top \;.$$

Every ascending or descending chain has a least upper bound and greatest lower bound.

Examples: lattices

1. $(2^{\mathbb{N}}, \subseteq)$ is a complete lattice with $\bigsqcup S = \bigcup_{M \in S} M$ for every subset $S \subseteq 2^{\mathbb{N}}$.
2. $(\mathbb{N}, \leqslant)$ is not a complete lattice, as, e.g., the chain $\mathbb{N}$ has no upper bound.
3. Which of the following structures are complete lattices?



where edge $u \rightarrow v$ means that $v \sqsubseteq u$

Overview

Monotonicity

Monotonicity

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be partial orders, and let $F : D \to D'$. F is called monotonic (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every $d_1, d_2 \in D$,

$$d_1 \sqsubseteq d_2 \quad \text{implies} \quad F(d_1) \sqsubseteq' F(d_2).$$

Interpretation: monotonic functions preserve information

1. Let $T := \{S \subseteq \mathbb{N} \mid S \text{ finite}\}$. Then $F_1 : T \to \mathbb{N} : S \mapsto \sum_{n \in S} n$ is monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$ and $(\mathbb{N}, \leqslant)$.
2. $F_2 : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : S \mapsto \mathbb{N} \setminus S$ is not monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$ (since, e.g., $\varnothing \subseteq \mathbb{N}$ but $F_2(\varnothing) = \mathbb{N} \not\subseteq F_2(\mathbb{N}) = \varnothing$).

Monotonicity on chains

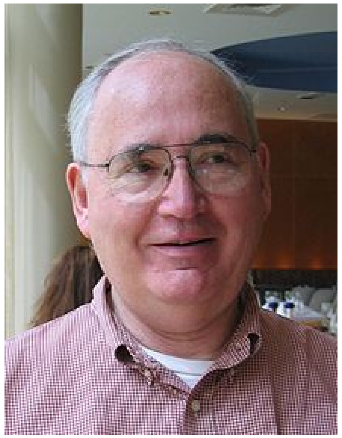
The following lemma states how chains behave under monotonic functions.

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be complete lattices, $F : D \to D'$ monotonic, and $S \subseteq D$ a chain in D. Then:

1. $F(S) := \{ F(d) \mid d \in S \}$ is a chain in D'.
2. $\bigsqcup F(S) \sqsubseteq' F(\bigsqcup S)$.

Proof.

Left as a homework exercise. □



Dana Scott (1932–)

denotational semantics, NFA determinisation, Scott continuity ...

Scott continuity

A function F is continuous if applying F and taking LUBs is commutable:

Scott continuity

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be complete lattices and $F : D \to D'$. Then F is called continuous if, for every non-empty chain $S \subseteq D$,

$$F\left(\bigsqcup S\right) = \bigsqcup F(S).$$

Every continuous function is monotonic.

Proof.

1. Let $d \sqsubseteq e$. Then $\{ d, e \}$ is a chain with $\bigsqcup \{ d, e \} = e$.
2. Let $F : D \to D'$ for complete lattice D'. Then $F(d) \sqsubseteq' \bigsqcup \{ F(d), F(e) \}$.
3. Let F be continuous. Then: $\bigsqcup \{ F(d), F(e) \} = F(\bigsqcup \{ d, e \}) = F(e)$

□

Overview

Fixed points

Fixed point

Let $F : D \to D$ be a function. Element $d \in D$ is called a fixed point of F if and only if $F(d) = d$.

Examples

1. Function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^2 - 3x + 4$ has a fixed point at 2.
2. For function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x$, all $x \in \mathbb{R}$ are fixed points.
3. Function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x + 4$ has no fixed points.
4. Function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = \frac{x}{2} + \frac{1}{x}$ has a fixed point at $\sqrt{2}$.
5. Function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = \cos(x)$ has a fixed point, but this is hard to determine.

How to find fixed points?

Naive scheme: start with an initial value x_0, and then iterate: $x_{n+1} = f(x_n)$.

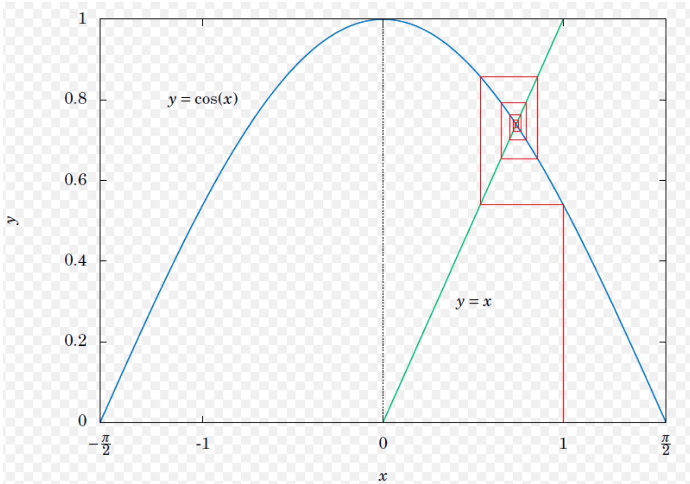
(Wrong) idea: as n grows larger, x_n converges to some fixed point of F.

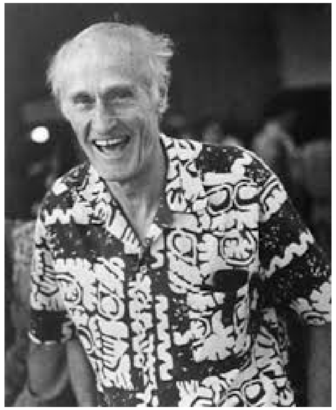
1. Take $F(x) = \frac{x}{2} + \frac{1}{x}$ and $x_0 = 1$. Iterations yields: $\frac{3}{2}, \frac{17}{12}, \frac{17}{24} + \frac{12}{17}$ which indeed approximates $\sqrt{2}$.
2. Take $F(x) = \frac{5}{2}x - \frac{3}{2}x^2$. Iteration converges to 1.
3. But, take $F(x) = \frac{13}{4}x - \frac{3}{2}x^2$. Iteration oscillates between two points, regardless of the initial value x_0.

When does such an iterative scheme yield (i.e., approximate) a fixed point, and if so, which fixed point?

We consider this for complete lattices and continuous functions.

Iterative scheme to determine a fixed point





Stephen Kleene (1909–1994)

regular expressions, recursion theory, fixed point theory, ...

Kleene's fixpoint theorem

Kleene's fixpoint theorem

Let $(D, \sqsubseteq)$ be a complete lattice and $F : D \to D$ continuous. Then F has a least fixed point $\mathrm{lfp}\, F$ and greatest fixed point $\mathrm{gfp}\, F$ respectively, given by:

$$\mathrm{lfp}\, F := \bigsqcup_{n \in \mathbb{N}} F^n(\bot) \quad \text{and} \quad \mathrm{gfp}\, F := \bigsqcap_{n \in \mathbb{N}} F^n(\top)$$

where $F^0(d) = d$ and $F^{n+1}(d) = F(F^n(d))$.

Proof.

on the board □

Examples

- Domain: complete lattice $(2^{\mathbb{N}}, \subseteq)$ with $\bigsqcup S = \bigcup_{N \in S} N$
- Function: $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : N \mapsto N \cup A$ for some fixed $A \subseteq \mathbb{N}$
  - F monotonic: $M \subseteq N \implies F(M) = M \cup A \subseteq N \cup A = F(N)$
  - F continuous: $F(\bigsqcup S) = F(\bigcup_{N \in S} N) = (\bigcup_{N \in S} N) \cup A = \bigcup_{N \in S} (N \cup A) = \bigcup_{N \in S} F(N) = \bigsqcup F(S)$.
- Fixpoint iteration: $N_n := F^n(\bigsqcup \varnothing)$ where $\bigsqcup \varnothing = \varnothing$
  - $N_0 = \bigsqcup \varnothing = \varnothing$
  - $N_1 = F(N_0) = \varnothing \cup A = A$
  - $N_2 = F(N_1) = A \cup A = A = N_n$ for every $n \geq 1$

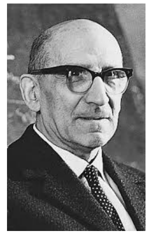
  $\Rightarrow \mathrm{lfp}\, F = A$
- Alternatively: $F(N) = N \cap A$

  $\Rightarrow \mathrm{lfp}\, F = \varnothing$, the least element $N \subseteq \mathbb{N}$ w.r.t. $\subseteq$ with $N \cap A = N$.

Knaster-Tarski theorem

Alfred Tarski (1901–1983)

Bronislaw Knaster (1893–1990)



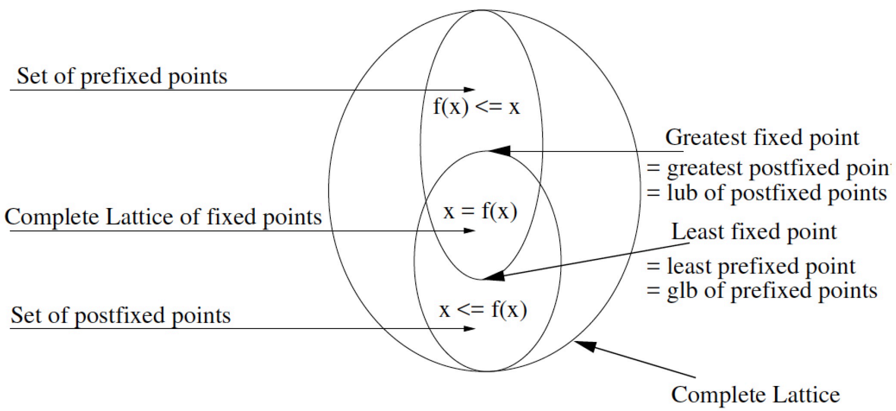
Knaster-Tarski theorem

Every monotonic function F on complete lattice $(D, \sqsubseteq)$ has a complete lattice of fixed points.

Proof.

For monotonic F on D, we have $\{ F(d) \mid d \in D \wedge F(d) = d \}$ is a complete lattice. On the black board. □

Pictorially



Park's lemma

Park's lemma

Let $(D, \sqsubseteq)$ be a complete lattice, $d \in D$, and let $F : D \to D$ be monotonic. Then:

1. $F(d) \sqsubseteq d$ implies $\text{lfp}\, F \sqsubseteq d$, and (pre-fixed point)
2. $d \sqsubseteq F(d)$ implies $d \sqsubseteq \text{gfp}\, F$ (post-fixed point)

Take-home messages

- Complete lattices are posets where every subset has a sup (and inf)
- Monotonic functions preserve the ordering of elements
- Functions are continuous if applying them and taking lub's commute
- Continuous function F on a complete lattice has an lfp (and gfp)
  which can be obtained by iterating F from below (or above)
- Every monotonic function on a complete lattice has a complete lattice of fixed points

Next part: Dijkstra's weakest pre-conditions

Next lecture

Thursday Nov 10, 16:30