Probabilistic Programming

Lecture #6: Syntax and Operational Semantics of pGCL

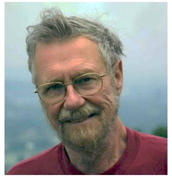
Joost-Pieter Katoen

RWTH Lecture Series on Probabilistic Programming 2022-23

Overview

Overview

Dijkstra's guarded command language: Syntax



- `skip` — empty statement
- `diverge` — divergence
- `x := E` — assignment
- `prog1 ; prog2` — sequential composition
- `if (G) prog1 else prog2` — choice
- `prog1 [] prog2` — non-deterministic choice
- `while (G) prog` — iteration

Elementary `pGCL` ingredients

- Program variables $x \in \mathit{Vars}$ whose values are fractional numbers
- Arithmetic expressions E over the program variables
- Boolean expressions G (aka: guards) over the program variables
- Distribution expressions $\mu : \Sigma \to \mathit{Dist}(\mathbb{Q})$
- Probability expressions $p : \Sigma \to [0, 1] \cap \mathbb{Q}$

where Σ is the set of program states; made precise later

Probabilistic GCL: Syntax

Dexter Kozen

Annabelle McIver

Carroll Morgan

- **`skip`** — empty statement
- **`diverge`** — divergence
- `x := E` — assignment
- `x :r= mu` — **random assignment** ($x :\approx \mu$)
- `prog1 ; prog2` — sequential composition
- **`if`** `(G) prog1` **`else`** `prog2` — choice
- `prog1 [p] prog2` — **probabilistic choice**
- **`while`** `(G) prog` — iteration

Conditioning in the form of observe-statements omitted for now.

Let's start simple

```
x := 0  [0.5]  x := 1;
y := -1 [0.5]  y := 0
```

This program admits four runs and yields the outcome:

$$Pr[x=0, y=0] \;=\; Pr[x=0, y=-1] \;=\; Pr[x=1, y=0] \;=\; Pr[x=1, y=-1] \;=\; 1/4$$

A loopy program

For $0 < p < 1$ an arbitrary probability:

```
bool c := true;
int i := 0;
while (c) {
    i++;
    (c := false [p] c := true)
}
```

The loopy program models a geometric distribution with parameter p.

$$Pr[i = N] \;=\; (1-p)^{N-1} \cdot p \quad \text{for } N > 0$$

On termination

```
bool c := true;
int i := 0;
while (c) {
    i++;
    (c := false [p] c := true)
}
```

This program does not always terminate. It almost surely terminates.

The good, the bad, and the ugly

Duelling cowboys

```
int cowboyDuel(float a, b) {
  int t := A [0.5] t := B;
  bool c := true;
  while (c) {
    if (t = A) {
      (c := false [a] t := B);
    } else {
      (c := false [b] t := A);
    }
  }
  return t;
}
```


Random assignments

The random assignment $x :\approx \mu$ works as follows:

1. evaluate distribution expression μ in the current program state s
2. sample from the resulting probability distribution $\mu(s)$

   this yields the value v with probability $\mu(s)(v)$
3. assign the value v to the variable x.

For denoting distribution expressions, we use the bra–ket notation.

$$\frac{1}{2} \cdot [a\rangle + \frac{1}{3} \cdot [b\rangle + \frac{1}{6} \cdot [c\rangle$$

denotes the distribution μ with $\mu(a) = 1/2$, $\mu(b) = 1/3$, and $\mu(c) = 1/6$. The support set of μ equals $\{ a, b, c \}$

Examples on the black board.

Overview

1. Probabilistic Guarded Command Language
2. Operational semantics of pGCL
3. Expected Rewards
4. Recursion

Why formal semantics matters

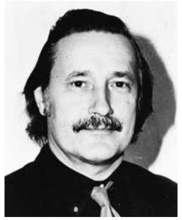
- Unambiguous meaning to all programs
- Basis for proving correctness
  - of programs
  - of program transformations
  - of program equivalence
  - of static analysis
  - of compilers
  - ......

The inventors of semantics

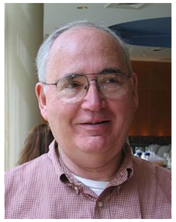
Tony Hoare

Robert W. Floyd

Gordon Plotkin



Christopher Strachey



Dana Scott

Approaches to semantics

- Operational semantics: (developed by Plotkin)
  - The meaning of a program = how it executes on an abstract machine.
  - Useful for modelling the execution behaviour of a program.
- Axiomatic semantics: (developed by Floyd and Hoare)
  - Provides correctness assertions for each program construct.
  - Useful for verifying that the program's computed results are correct with respect to the specification.
- Denotational semantics: (developed by Strachey and Scott)
  - Provides a mapping of language constructs onto mathematical objects.
  - Useful for obtaining an abstract insight into the working of a program.

Today: operational semantics of `pGCL` in terms of Markov chains.

Later: denotational semantics of `pGCL` in terms of weakest preconditions.

Structural operational semantics: ingredients

- Variable valuation $s : Vars \to \mathbb{Q}$ maps each program variable onto a value (here: rational numbers)
- Variable update: for variable x and value $v \in \mathbb{Q}$, let

$$s[x := v](y) = s(y) \quad \text{if } x \neq y \quad \text{and} \quad s[x := v](y) = v \text{ otherwise.}$$

- Let $[\![E]\!]$ denote the valuation of expression E
- Program configuration (aka: state) $\langle P, s\rangle$ denotes that
  - program P is next to be executed (aka: program counter), and
  - the current variable valuation equals s.
- Transition rules for the execution of commands: $\langle P, s\rangle \longrightarrow \langle P', s'\rangle$

  denoted as $\dfrac{\text{premise}}{\text{conclusion}}$ where the premise is omitted if it equals true.

Operational semantics

Aim: Model the behaviour of a program P by the MC $[\![P]\!]$.

This MC is defined using structured operational semantics

Operational semantics

Aim: Model the behaviour of a program P by the MC $[\![P]\!]$.

Approach:

- Take states of the form
  - $\langle Q, s\rangle$ with program Q or $\downarrow$, and variable valuation $s : Var \to \mathbb{Q}$
  - $\langle sink\rangle$ models program termination (successful)
- Take initial state $\langle P, s\rangle$ where s fulfils the initial conditions
- Take transition relation $\to$ as smallest relation satisfying the transition rules

Transition rules (1)

$$\langle \texttt{skip}, s\rangle \to \langle \downarrow, s\rangle$$

$$\langle \downarrow, s\rangle \to \langle sink\rangle \qquad \langle sink\rangle \to \langle sink\rangle$$

$$\langle x := E, s\rangle \to \langle \downarrow, s[x := s([\![E]\!])]\rangle$$

$$\frac{\mu(s)(v) = a > 0}{\langle x :\approx \mu, s\rangle \xrightarrow{a} \langle \downarrow, s[x := v]\rangle}$$

$$\langle P[\,p\,]\,Q, s\rangle \to \mu \text{ with } \mu(\langle P, s\rangle) = p \text{ and } \mu(\langle Q, s\rangle) = 1-p$$

Random assignments

The random assignment $x :\approx \mu$ works as follows:

1. evaluate distribution expression μ in the current program state s
2. sample from the resulting probability distribution $\mu(s)$

   this yields the value v with probability $\mu(s)(v)$
3. assign the value v to the variable x.

For denoting distribution expressions, we use the bra–ket notation.

$$\frac{1}{2} \cdot [a\rangle + \frac{1}{3} \cdot [b\rangle + \frac{1}{6} \cdot [c\rangle$$

denotes the distribution μ with $\mu(a) = 1/2$, $\mu(b) = 1/3$, and $\mu(c) = 1/6$. The support set of μ equals $\{ a, b, c \}$

Examples on the black board.

Transition rules (2)

$$\frac{\langle P, s\rangle \to \mu}{\langle P; Q, s\rangle \to \nu} \text{ with } \nu(\langle P'; Q', s'\rangle) = \mu(\langle P', s'\rangle) \text{ where } \downarrow; Q = Q$$

$$\frac{s \models G}{\langle \text{if } (G)\{P\} \text{ else } \{Q\}, s\rangle \to \langle P, s\rangle} \qquad \frac{s \not\models G}{\langle \text{if } (G)\{P\} \text{ else } \{Q\}, s\rangle \to \langle Q, s\rangle}$$

$$\frac{s \models G}{\langle \text{while}(G)\{P\}, s\rangle \to \langle P; \text{while } (G)\{P\}, s\rangle} \qquad \frac{s \not\models G}{\langle \text{while}(G)\{P\}, s\rangle \to \langle \downarrow, s\rangle}$$

Example

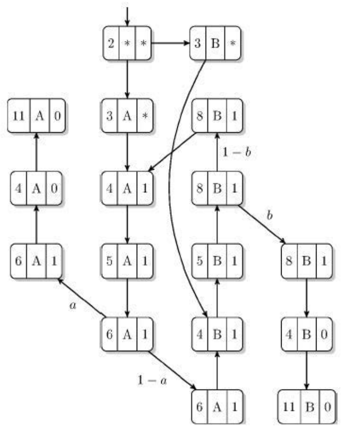
```
x := 0 [1/3] x := 1;
y := unif[1..5]
```

```
x := unif[1..5];
if (x >= 2) { y := unif[1..x] } else { y := x }
```

```
x +:= 1 [1/(abs(x)+1)] x -:= 1
```


Duelling cowboys

```
int cowboyDuel(float a, b) {
  int t := A [0.5] t := B;
  bool c := true;
  while (c) {
    if (t = A) {
      (c := false [a] t := B);
    } else {
      (c := false [b] t := A);
    }
  }
  return t;
}
```



This (parametric) MC is finite. Once we count the number of shots before one of the cowboys dies, the MC becomes countably infinite.

Duelling cowboys

```
int cowboyDuel(float a, b) { // 0 < a < 1, 0 < b < 1
   int t := A [0.5] t := B; // decide who shoots first
   bool c := true;
   while (c) {
     if (t = A) {
         (c := false [a] t := B); // A shoots B with prob. a
     } else {
         (c := false [b] t := A); // B shoots A with prob. b
     }
   }
return t; // the survivor
}
```

Claim:

Cowboy A wins the duel with probability $\frac{(1-b)\cdot\frac{1}{2}a}{a+b-a\cdot b}$.

Proof

The outcome of a pGCL program

Unlike a deterministic program, a pGCL program P has not a deterministic output for a given input. Instead, it yields a unique probability distribution over its final states.

In fact, this is a sub-distribution (probability mass at most one), as with a (possibly positive) probability, P may diverge.

Let P be a pGCL program and s an input state. Then the distribution over final states obtained by running P starting in s is given by $Pr(s \models \Diamond \langle\downarrow, \cdot\rangle)$.

If P is a program whose MC is finite-state, then $Pr(s \models \Diamond \langle\downarrow, \cdot\rangle)$ can be determined by solving a linear equation system.

Overview

Rewards

To reason about resource usage in MCs: use rewards.

A reward MC is a pair (D, r) with D an MC with state space Σ and $r : \Sigma \to \mathbb{R}$ a function assigning a real reward to each state.

The reward $r(\sigma)$ stands for the reward earned on leaving state σ.

Let $\pi = \sigma_0 \ldots \sigma_n$ be a finite path in (D, r) and $G \subseteq \Sigma$ a set of target states with $\pi \in \Diamond G$. The cumulative reward along π until reaching G is:

$$r_G(\pi) = r(\sigma_0) + \ldots + r(\sigma_{k-1}) \text{ where } \sigma_i \notin G \text{ for all } i < k \text{ and } \sigma_k \in G.$$

If $\pi \notin \Diamond G$, then $r_G(\pi) = \infty$.

Expected reward for reachability

Let σ be such that $Pr(\sigma \models \Diamond G) = 1$.

Then: the expected reward until reaching $G \subseteq \Sigma$ from $\sigma \in \Sigma$ is:

$$\mathrm{ER}(\sigma, \Diamond G) \ = \ \sum_{\widehat{\pi}} Pr(\widehat{\pi}) \cdot r_G(\widehat{\pi})$$

where $\widehat{\pi} = \sigma_0 \ldots \sigma_k$ is such that $\sigma_k \in G$, $\sigma_0 = \sigma$ and $\sigma_i \notin G$ for all $i < k$.

If $Pr(\sigma \models \Diamond G) < 1$, then let $\mathrm{ER}(\sigma, \Diamond G) = \infty$.

On computing expected rewards

Expected rewards in finite Markov chains can be computed in polynomial time by solving a system of linear equations.
(details on the black board.)

Equation system for expected rewards

Overview

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4. Recursion

Probabilistic GCL with recursion: Syntax

- `skip` — empty statement
- `x := E` — assignment
- `x :r= mu` — random assignment ($x :\approx \mu$)
- `prog1 ; prog2` — sequential composition
- `if (G) prog1 else prog2` — choice
- `prog1 [p] prog2` — probabilistic choice
- `while (G) prog` — iteration
- `proc P = prog` — **process definition**
- `call P` — **process invocation**

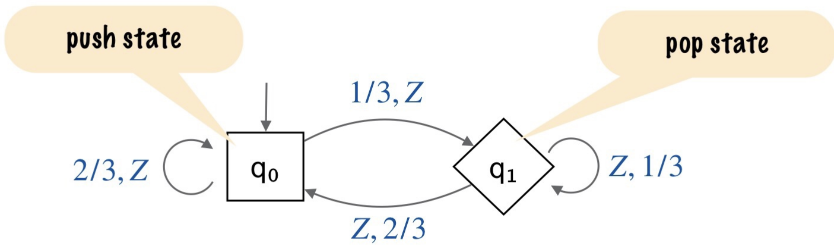
Recursion does not increase the expressive power, but is often convenient.

Pushdown Markov chains

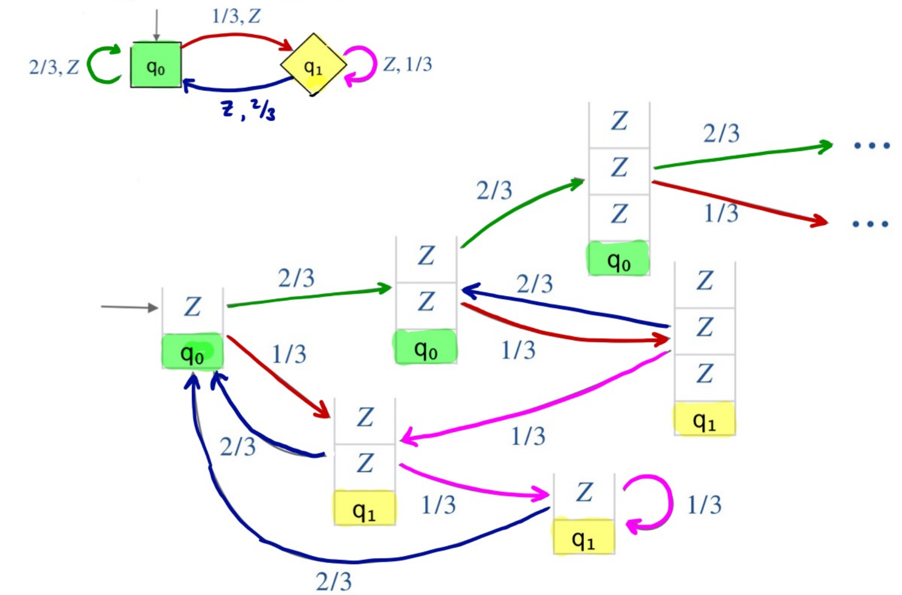
A pushdown Markov chain consists of:

- $\Sigma = (\Sigma_{push}, \Sigma_{pop}, \Sigma_{int})$, a finite set of states
- Γ is a finite stack alphabet
- $(\sigma_I, Z_0) \in \Sigma \times \Gamma$, the initial configuration
- probabilistic transition functions:
  - $P_{push} : \Sigma_{push} \to Dist(\Sigma \times \Gamma)$ — push transitions
  - $P_{pop} : \Sigma_{pop} \times \Gamma \to Dist(\Sigma)$ — pop transitions
  - $P_{int} : \Sigma_{int} \to Dist(\Sigma)$ — internal transitions

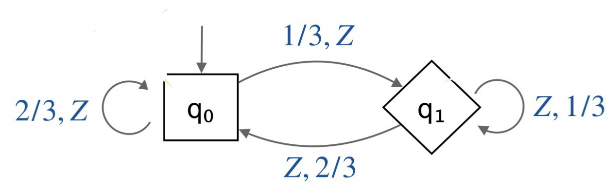
Pushdown MC = Markov chain + stack

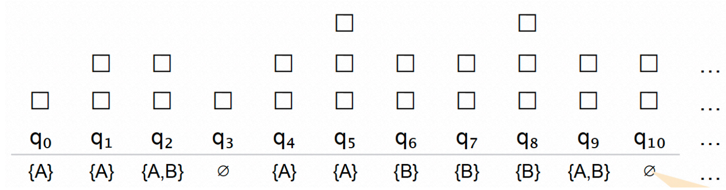


Configuration graph



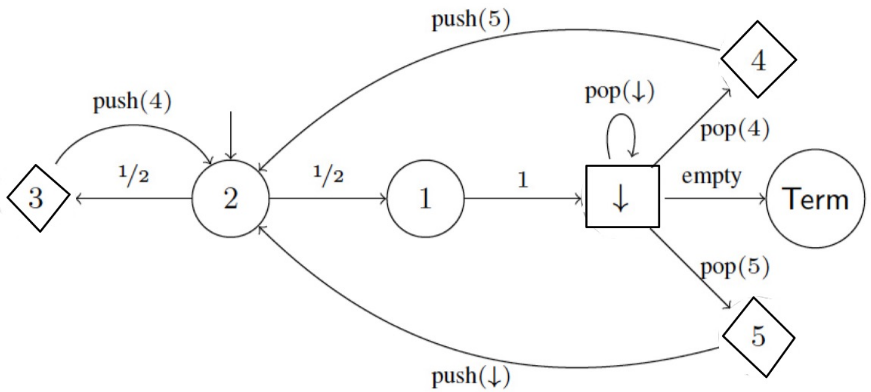
Runs of a pushdown MC





Assuming states are labeled with sets of atomic propositions

From programs to pushdown MCs



{skip^1} $[1/2]^2$ {call P^3; call P^4; call P^5 }

Take-home messages

- pGCL is a "base" imperative probabilistic programming language
- Key ingredients: probabilistic choice and random assignments
- A pGCL program corresponds to a (countably infinite) Markov chain
- Computing expected rewards in finite MCs = solving linear equations
- Recursion can be added to pGCL and yields pushdown MCs

Next lecture

Tuesday Nov 8, 16:30

No lecture on Nov 3;

next exercise class Nov 4