Concurrency Theory

Winter Semester 2019/20

Lecture 6: Mutually Recursive Equational Systems

Joost-Pieter Katoen and Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

https://moves.rwth-aachen.de/teaching/ws-19-20/ct/

Software Modeling and Verification Chair
RWTH AACHEN UNIVERSITY

Outline of Lecture 6

The Fixed-Point Theorem

Alfred Tarski (1901–1983)

Theorem (Tarski's fixed-point theorem)

Let $(D, \sqsubseteq)$ be a complete lattice and $f : D \rightarrow D$ monotonic. Then f has a least fixed point $\mathrm{fix}(f)$ and a greatest fixed point $\mathrm{FIX}(f)$ given by

$$\mathrm{fix}(f) = \bigsqcap \{d \in D \mid f(d) \sqsubseteq d\} \qquad \textit{(GLB of all pre-fixed points of } f\textit{)}$$

$$\mathrm{FIX}(f) = \bigsqcup \{d \in D \mid d \sqsubseteq f(d)\} \qquad \textit{(LUB of all post-fixed points of } f\textit{)}$$

Proof.

on the board □

The Fixed-Point Theorem for Finite Lattices

Theorem (Fixed-point theorem for finite lattices)

Let $(D, \sqsubseteq)$ be a finite complete lattice and $f : D \to D$ monotonic. Then

$$\mathrm{fix}(f) = f^m(\bot) \quad \textit{and} \quad \mathrm{FIX}(f) = f^M(\top)$$

for some $m, M \in \mathbb{N}$ where $f^0(d) := d \quad \textit{and} \quad f^{k+1}(d) := f(f^k(d))$.

Proof.

on the board □

Application to HML with Recursion

Lemma

Let $(S, Act, \longrightarrow)$ *be an LTS and* $F \in HMF_X$. *Then*

1. $[\![F]\!] : 2^S \to 2^S$ *is monotonic w.r.t.* $(2^S, \subseteq)$
2. $\mathrm{fix}([\![F]\!]) = \bigcap\{T \subseteq S \mid [\![F]\!](T) \subseteq T\}$
3. $\mathrm{FIX}([\![F]\!]) = \bigcup\{T \subseteq S \mid T \subseteq [\![F]\!](T)\}$

If, in addition, S *is finite, then*

4. $\mathrm{fix}([\![F]\!]) = [\![F]\!]^m(\emptyset)$ *for some* $m \in \mathbb{N}$
5. $\mathrm{FIX}([\![F]\!]) = [\![F]\!]^M(S)$ *for some* $M \in \mathbb{N}$

Proof.

1. by induction on the structure of F (details omitted)
2. by Lemma 4.15 and Theorem 5.5
3. by Lemma 4.15 and Theorem 5.5
4. by Lemma 4.15 and Theorem 5.7
5. by Lemma 4.15 and Theorem 5.7 □

Outline of Lecture 6

Greatest Fixed Points and Invariants

- Invariants (cf. Example 4.5):
  - $Inv(F) \overset{max}{=} F \wedge [Act]Inv(F)$ for $F \in HMF$
  - $s \models Inv(F)$ if all states reachable from s satisfy F

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- Let $inv : 2^S \to 2^S : T \mapsto [\![F]\!] \cap [\cdot Act \cdot](T)$ be the corresponding semantic function
- By Lemma 5.9, $\mathrm{FIX}(inv) = \bigcup\{T \subseteq S \mid T \subseteq inv(T)\}$

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- Direct formulation of invariance property:

$$Inv = \{s \in S \mid \forall w \in Act^*, s' \in S : s \xrightarrow{w} s' \implies s' \in [\![F]\!]\}$$

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Theorem 6.1

For every LTS $(S, Act, \longrightarrow)$, $Inv = \mathrm{FIX}(inv)$ holds.

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Proof.

on the board □

Least Fixed Points and Possibilities

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  - $Pos(F) \overset{min}{=} F \vee \langle Act \rangle Pos(F)$
  - $s \models Pos(F)$ if a state satisfying F is reachable from s

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- Let $pos : 2^S \to 2^S : T \mapsto [\![F]\!] \cup \langle \cdot Act \cdot \rangle(T)$ be the corresponding semantic function
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Theorem 6.2

For every LTS $(S, Act, \longrightarrow)$, $Pos = \mathrm{fix}(pos)$ holds.

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Theorem 6.2

For every LTS $(S, Act, \longrightarrow)$, $Pos = \mathrm{fix}(pos)$ holds.

Proof.

similar to Theorem 6.1 □

Outline of Lecture 6

Recap: Fixed-Point Theory

Fixed Points and System Properties

Mutually Recursive Equational Systems

Mixing Least and Greatest Fixed Points

Introducing Several Variables

Sometimes necessary: using more than one variable

Example 6.3

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can be specified by

$$Inv(\langle a \rangle Forever(b))$$

where

$$\begin{aligned} Inv(F) &\stackrel{max}{=} F \wedge [Act]Inv(F) \qquad \text{(cf. Theorem 6.1)} \\ Forever(b) &\stackrel{max}{=} \langle b \rangle Forever(b) \end{aligned}$$

Syntax of Mutually Recursive Equational Systems

Definition 6.4 (Syntax of mutually recursive equational systems)

Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a set of variables. The set $HMF_{\mathcal{X}}$ of Hennessy-Milner formulae over $\mathcal{X}$ is defined by the following syntax:

$$
\begin{array}{rlr}
F ::= & X_i & \text{(variable)} \\
| & \mathrm{tt} & \text{(true)} \\
| & \mathrm{ff} & \text{(false)} \\
| & F_1 \wedge F_2 & \text{(conjunction)} \\
| & F_1 \vee F_2 & \text{(disjunction)} \\
| & \langle \alpha \rangle F & \text{(diamond)} \\
| & [\alpha] F & \text{(box)}
\end{array}
$$

where $1 \leq i \leq n$ and $\alpha \in Act$. A mutually recursive equational system has the form

$$(X_i = F_{X_i} \mid 1 \leq i \leq n)$$

where $F_{X_i} \in HMF_{\mathcal{X}}$ for every $1 \leq i \leq n$.

Semantics of Recursive Equational Systems I

As before: semantics of formula depends on states satisfying the variables

Definition 6.5 (Semantics of mutually recursive equational systems)

Let $(S, Act, \longrightarrow)$ be an LTS and $E = (X_i = F_{X_i} \mid 1 \leq i \leq n)$ a mutually recursive equational system. The semantics of E, $[\![E]\!] : (2^S)^n \to (2^S)^n$, is defined by

$$[\![E]\!](T_1, \ldots, T_n) := ([\![F_{X_1}]\!](T_1, \ldots, T_n), \ldots, [\![F_{X_n}]\!](T_1, \ldots, T_n))$$

where

$$\begin{aligned}
[\![X_i]\!](T_1, \ldots, T_n) &:= T_i \\
[\![\mathrm{tt}]\!](T_1, \ldots, T_n) &:= S \\
[\![\mathrm{ff}]\!](T_1, \ldots, T_n) &:= \emptyset \\
[\![F_1 \wedge F_2]\!](T_1, \ldots, T_n) &:= [\![F_1]\!](T_1, \ldots, T_n) \cap [\![F_2]\!](T_1, \ldots, T_n) \\
[\![F_1 \vee F_2]\!](T_1, \ldots, T_n) &:= [\![F_1]\!](T_1, \ldots, T_n) \cup [\![F_2]\!](T_1, \ldots, T_n) \\
[\![\langle\alpha\rangle F]\!](T_1, \ldots, T_n) &:= \langle\cdot\alpha\cdot\rangle([\![F]\!](T_1, \ldots, T_n)) \\
[\![[\alpha] F]\!](T_1, \ldots, T_n) &:= [\cdot\alpha\cdot]([\![F]\!](T_1, \ldots, T_n))
\end{aligned}$$

Semantics of Recursive Equational Systems II

Lemma 6.6

Let $(S, Act, \longrightarrow)$ *be a* *finite* *LTS and* $E = (X_i = F_{X_i} \mid 1 \leq i \leq n)$ *a mutually recursive equational system. Let* $(D, \sqsubseteq)$ *be given by* $D := (2^S)^n$ *and*

$$(T_1, \ldots, T_n) \sqsubseteq (T'_1, \ldots, T'_n)$$

iff $T_i \subseteq T'_i$ *for every* $1 \leq i \leq n$.

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1. $(D, \sqsubseteq)$ *is a complete lattice with*

$$\bigsqcup\{(T^i_1, \ldots, T^i_n) \mid i \in I\} = \left(\bigcup\{T^i_1 \mid i \in I\}, \ldots, \bigcup\{T^i_n \mid i \in I\}\right)$$
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Proof.

omitted □

A Mutually Recursive Specification

Example 6.7

$$\begin{tikzcd} s \arrow[r, bend left, "a"] & s_1 \arrow[l, bend left, "b"] & s_2 \arrow[l, "a"'] \arrow[r, "a"] & s_3 \arrow[loop right, "b"] \end{tikzcd}$$

- Let $S := \{s, s_1, s_2, s_3\}$ and E given by

$$\begin{aligned} X &= \langle a \rangle Y \wedge [a]Y \wedge [b]\mathrm{ff} \\ Y &= \langle b \rangle X \wedge [b]X \wedge [a]\mathrm{ff} \end{aligned}$$

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- Interpretation:
  - X: “has no b-successor and ≥ 1 a-successors that all satisfy Y”
  - Y: “has no a-successor and ≥ 1 b-successors that all satisfy X”

A Mutually Recursive Specification

Example 6.7

$$s \underset{b}{\overset{a}{\rightleftarrows}} s_1 \xleftarrow{a} s_2 \xrightarrow{a} s_3 \circlearrowleft b$$

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$\Rightarrow$ expected: $X = \{s\}$, $Y = \{s_1\}$

Mutually Recursive Equational Systems

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  $\Rightarrow$ expected: $X = \{s\}$, $Y = \{s_1\}$
- Computation of $\mathrm{FIX}([\![E]\!])$: on the board

Outline of Lecture 6

Mixing Least and Greatest Fixed Points I

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- **But:** too restrictive

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Example 6.8

"It is possible for the system to reach a state which has a livelock (i.e., an infinite sequence of internal steps)."

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Example 6.8

"It is possible for the system to reach a state which has a livelock (i.e., an infinite sequence of internal steps)."

can be specified by

$$Pos(Livelock)$$

where

$$\begin{aligned} Pos(F) &\stackrel{min}{=} F \vee \langle Act \rangle Pos(F) \qquad \text{(cf. Theorem 6.2)} \\ Livelock &\stackrel{max}{=} \langle \tau \rangle Livelock \end{aligned}$$

(thus, $Livelock \equiv Forever(\tau)$ [cf. Example 6.3])

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Caveat: arbitrary mixing can entail non-monotonic behaviour

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Fixed-point iteration:

$$(\bot, \top) = (\emptyset, S)$$

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Solution: nesting of specifications by partitioning equations into a sequence of blocks such that all equations in one block

- are of same type (either *min* or *max*) and
- use only variables defined in the same or subsequent blocks

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Solution: nesting of specifications by partitioning equations into a sequence of blocks such that all equations in one block

- are of same type (either *min* or *max*) and
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$\Longrightarrow$ bottom-up, block-wise evaluation by fixed-point iteration

Mixing Least and Greatest Fixed Points III

Example 6.10 (cf. Example 6.8)

$$
\begin{aligned}
\mathit{PosLL} &\overset{\mathit{min}}{=} \mathit{Livelock} \vee \langle \mathit{Act} \rangle \mathit{PosLL} \\
\mathit{Livelock} &\overset{\mathit{max}}{=} \langle \tau \rangle \mathit{Livelock}
\end{aligned}
$$

$$
\begin{tikzcd}
s \arrow[r, "a"] & p \arrow[r, "\tau"] \arrow[loop below, "\tau"] & q \arrow[r, "\tau"] & r
\end{tikzcd}
$$

Mixing Least and Greatest Fixed Points III

Example 6.10 (cf. Example 6.8)

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PosLL &\overset{min}{=} Livelock \vee \langle Act \rangle PosLL \\
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$$
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s \arrow[r, "a"] & p \arrow[r, "\tau"] \arrow[loop below, "\tau"] & q \arrow[r, "\tau"] & r
\end{tikzcd}
$$

1. Greatest fixed-point iteration for $Livelock : T \mapsto \langle \cdot \tau \cdot \rangle (T)$:

$$\top = S = \{s, p, q, r\}$$

Mixing Least and Greatest Fixed Points III

Example 6.10 (cf. Example 6.8)

$$PosLL \overset{min}{=} Livelock \vee \langle Act \rangle PosLL$$
$$Livelock \overset{max}{=} \langle \tau \rangle Livelock$$

$$\begin{tikzcd} s \arrow[r, "a"] & p \arrow[r, "\tau"] \arrow[loop below, "\tau"] & q \arrow[r, "\tau"] & r \end{tikzcd}$$

1. Greatest fixed-point iteration for $Livelock : T \mapsto \langle \cdot \tau \cdot \rangle (T)$:

$$\top = S = \{s, p, q, r\} \mapsto \{p, q\}$$

Mixing Least and Greatest Fixed Points III

Example 6.10 (cf. Example 6.8)

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\begin{aligned}
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2. Least fixed-point iteration for $PosLL : T \mapsto \{p\} \cup \langle \cdot Act \cdot \rangle (T)$:

$$\bot = \emptyset$$

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Mixing Least and Greatest Fixed Points

The Modal μ-Calculus

- Logic that supports free mixing of least and greatest fixed points:
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- Overview paper:
  - O. Burkart, D. Caucal, F. Moller, B. Steffen: *Verification on Infinite Structures*, Chapter 9 of *Handbook of Process Algebra*, Elsevier, 2001, 545–623