Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

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Overview

Negative exponential distribution

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Density of exponential distribution

The density of an *exponentially distributed* r.v. Y with *rate* $\lambda \in \mathbb{R}_{>0}$ is:

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- Variance $Var[Y] = \int_0^\infty (x - E[X])^2 \lambda \cdot e^{-\lambda \cdot x} \, dx \;=\; \frac{1}{\lambda^2}$

Continuous-time Markov chain

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A CTMC is a tuple $(S, \mathbf{P}, r, \iota_{\text{init}}, AP, L)$ where

- $(S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ is a DTMC, and
- $r : S \to \mathbb{R}_{>0}$, the exit-rate function

Let $\mathbf{R}(s, s') = \mathbf{P}(s, s') \cdot r(s)$ be the transition rate of transition (s, s')

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- residence time in state s is exponentially distributed with rate $r(s)$.
- phrased alternatively, the average residence time of state s is $\frac{1}{r(s)}$.

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The probability to *take some* outgoing transition from s in $[0, t]$ is:

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Proof:

On the blackboard.

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But: numerical instability due to fill-in of $(\mathbf{R}-\mathbf{r})^i$ in presence of positive and negative entries in the matrix $\mathbf{R}-\mathbf{r}$.

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It follows that $\overline{\mathbf{P}}$ is a stochastic matrix and $unif(r, \mathcal{C})$ is a CTMC.

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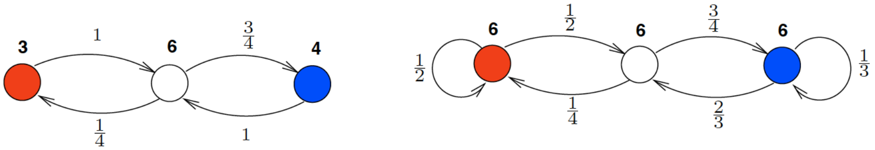

CTMC $\mathcal{C}$ and its uniformized counterpart $unif(6, \mathcal{C})$

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  2. decrease the transition probabilities by a factor $\frac{r(s)}{r}$, and
  3. increase the self-loop probability by a factor $\frac{r - r(s)}{r}$

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Let $r \in \mathbb{R}_{>0}$ such that $r \geqslant max_{s \in S}\ r(s)$. Then $unif(r, \mathcal{C}) = (S, \overline{\mathbf{P}}, \overline{r}, \iota_{\text{init}}, AP, L)$ with $\overline{r}(s) = r$ for all $s \in S$, and:

$$\overline{\mathbf{P}}(s, s') = \frac{r(s)}{r} \cdot \mathbf{P}(s, s') \text{ if } s' \neq s \quad \text{and} \quad \overline{\mathbf{P}}(s, s) = \frac{r(s)}{r} \cdot \mathbf{P}(s, s) + 1 - \frac{r(s)}{r}.$$

Intuition

- Fix all exit rates to (at least) the maximal exit rate r occurring in CTMC $\mathcal{C}$.
- Thus, $\frac{1}{r}$ is the shortest mean residence time in the CTMC $\mathcal{C}$.
- Then normalize the residence time of all states with respect to r as follows:
  1. replace an average residence time $\frac{1}{r(s)}$ by a shorter (or equal) one, $\frac{1}{r}$
  2. decrease the transition probabilities by a factor $\frac{r(s)}{r}$, and
  3. increase the self-loop probability by a factor $\frac{r - r(s)}{r}$

  That is, slow down state s whenever $r(s) < r$.

Overview

Strong bisimulation on DTMCs

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Probabilistic bisimulation [Larsen & Skou, 1989]

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For states in R, the conditional probability of moving by a single transition to another equivalence class is equal. In addition, either all states in an equivalence class C almost surely stay there, or have an option to escape from C.

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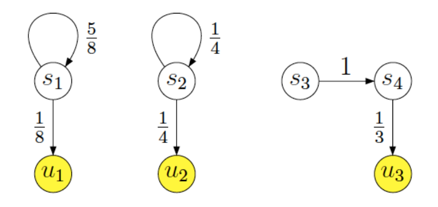

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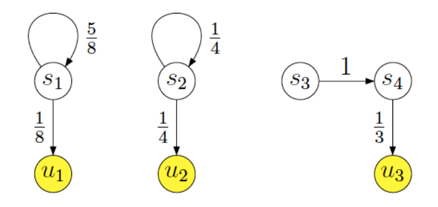

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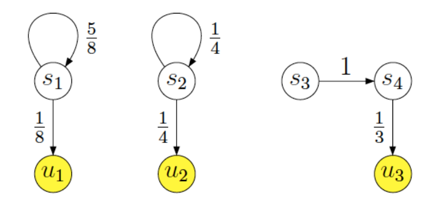

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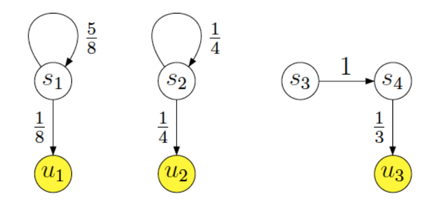

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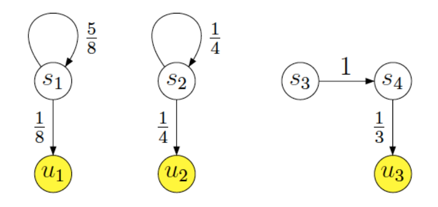

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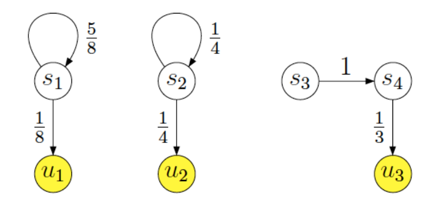

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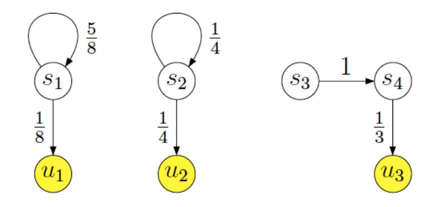

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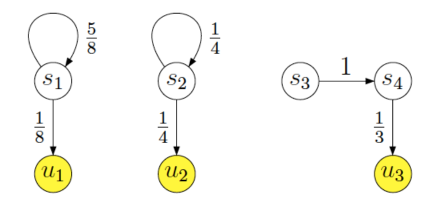

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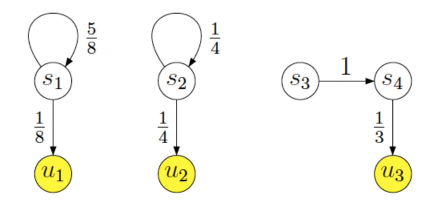

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Note that $\mathbf{P}(s_3, [s_3]_R) = 1$.

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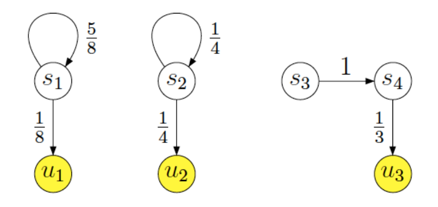

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$$\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, [s_1])} = \frac{1/8}{1{-}5/8} = \frac{1/4}{1{-}1/4} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, [s_2])} = \frac{1/3}{1} = \frac{\mathbf{P}(s_4, C)}{1 - \mathbf{P}(s_4, [s_4])}.$$

Note that $\mathbf{P}(s_3, [s_3]_R) = 1$. Since s_3 can reach a state outside $[s_3]$ as s_1, s_2 and s_4, it follows that $s_1 \approx_p s_2 \approx_p s_3 \approx_p s_4$.

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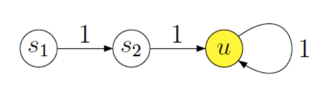

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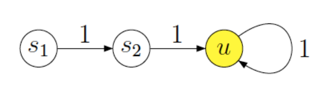

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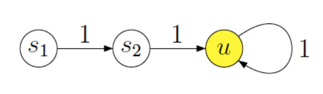

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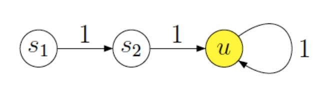

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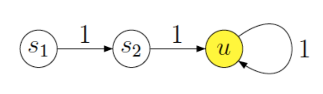

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Weak bisimulation on CTMCs

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Let $\mathcal{C}$ be a CTMC and s, t states in $\mathcal{C}$. Then: s is *weak probabilistically bisimilar* to t, denoted $s \approx_m t$, if there exists a weak probabilistic bisimulation R with $(s, t) \in R$.

A useful lemma

Let $\mathcal{C}$ be a CTMC and R an equivalence relation on S with $(s, t) \in R$, $\mathbf{P}(s, [s]_R) < 1$ and $\mathbf{P}(t, [t]_R) < 1$.

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Proof:

Left as an exercise.

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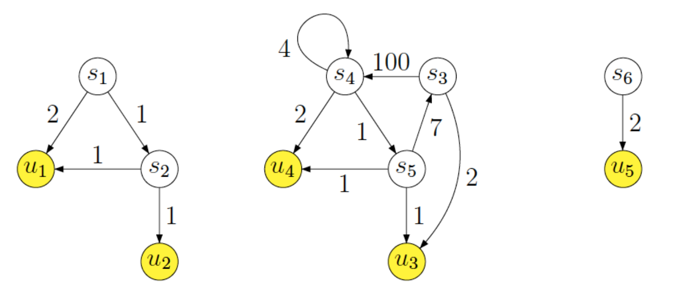

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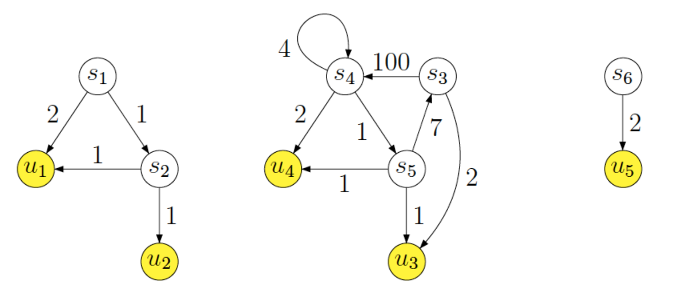

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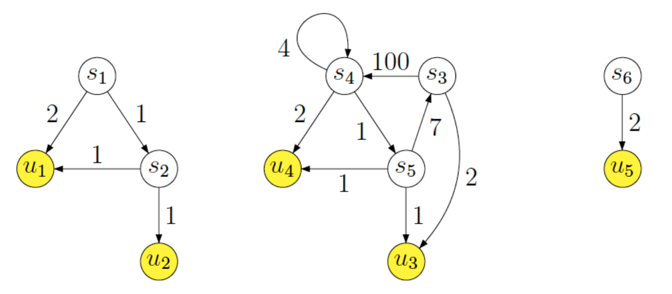

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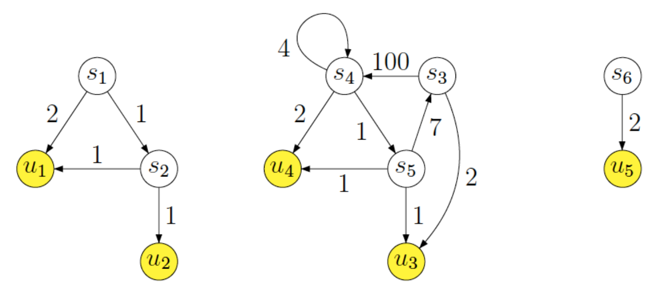

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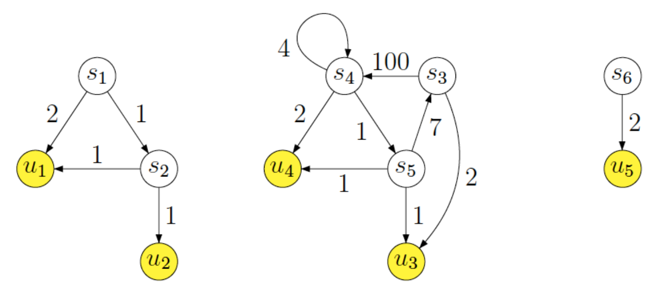

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Properties (without proof)

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For all uniform CTMCs $\mathcal{C}$ and states s, u in $\mathcal{C}$, we have:

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Preservation of transient probabilities

For all CTMCs $\mathcal{C}$ with states s, u in $\mathcal{C}$ and $t \in \mathbb{R}_{\geqslant 0}$, we have:

$$s \approx_m u \quad \text{implies} \quad \underline{p}^s(t) = \underline{p}^u(t)$$

where $\underline{p}^s(0) = \mathbf{1}_s$ and $\underline{p}^u(0) = \mathbf{1}_u$ where $\mathbf{1}_s$ is the characteristic function for state s, i.e., $\mathbf{1}_s(s') = 1$ iff $s = s'$.

Overview

Computing transient probabilities

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The transient probability vector $\underline{p}(t) = (p_{s_1}(t), \ldots, p_{s_k}(t))$ satisfies:

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As $\overline{\mathbf{P}}$ is a stochastic matrix, computing the matrix exponential $\overline{\mathbf{P}}^i$ is numerically stable.

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where r is the mean of the Poisson distribution.

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Remark

The Poisson distribution can be derived as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed.

Transient probabilities: example

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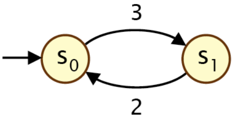

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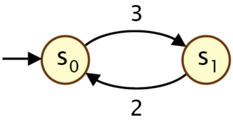

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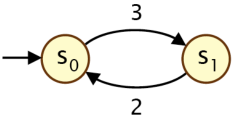

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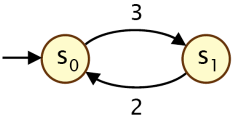

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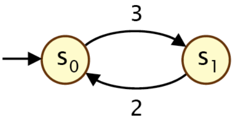

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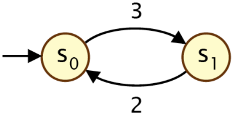

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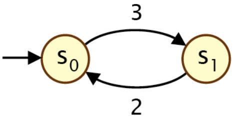

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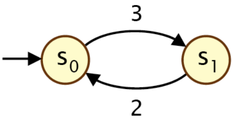

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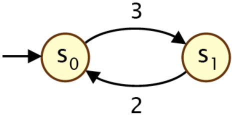

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$\sum_{i=0}^{\infty} e^{-rt} \frac{(rt)^i}{i!} = 1$ due to the fact that $e^{-rt} \frac{(rt)^i}{i!}$ is a (Poisson) distribution

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