Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

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  - ranges over all possible resolutions of nondeterminism.

PCTL syntax [Bianco & De Alfaro, 1995]

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  where $a \in AP$, φ is a path formula and $J \subseteq [0,1]$, $J \neq \varnothing$ is a non-empty interval.

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  Abbreviate $\mathbb{P}_{[0,0.5]}(\varphi)$ by $\mathbb{P}_{\leqslant 0.5}(\varphi)$ and $\mathbb{P}_{]0,1]}(\varphi)$ by $\mathbb{P}_{>0}(\varphi)$.

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- $s_0 \alpha_0 s_1 \alpha_1 s_2 \alpha_2 \ldots \models \Phi \, \mathsf{U}^{\leqslant n} \, \Psi$ if Φ holds until Ψ holds within n steps (where $s_i \alpha_{i+1}$ is a single step).

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- $s \models \mathbb{P}_J(\varphi)$ if the probability under all policies that paths starting in s fulfill φ lies in J.

Overview

Markov decision process (MDP)

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Assumption: in each state at least one action is enabled.

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Notation

$\mathcal{M}, s \models \Phi$ if and only if state-formula Φ holds in state s of (possibly infinite) MDP $\mathcal{M}$. As $\mathcal{M}$ is known from the context we simply write $s \models \Phi$.

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Satisfaction relation for state formulas

The satisfaction relation $\models$ is defined for PCTL state formulas by:

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\begin{array}{lll}
s \models a & \text{iff} & a \in L(s) \\
s \models \neg\Phi & \text{iff} & \text{not } (s \models \Phi) \\
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s \models \mathbb{P}_J(\varphi) & \text{iff} & \text{for all policies } \mathfrak{S} \text{ on } \mathcal{M}.\ Pr^{\mathfrak{S}}(s \models \varphi) \in J
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where $Pr^{\mathfrak{S}}(s \models \varphi) = Pr_s^{\mathfrak{S}}\{\pi \in Paths(s) \mid \pi \models \varphi\}$.

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$$s \models \mathbb{P}_{\leqslant p}(\varphi) \quad \text{iff} \quad Pr^{\max}(s \models \varphi) \leqslant p \quad \text{iff} \quad \sup_{\mathfrak{S}} Pr^{\mathfrak{S}}(s \models \varphi) \leqslant p$$

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as for any finite MDP an fm-policy exists that maximises or minimises φ.

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There is indeed no difference with the PCTL semantics for DTMC paths.

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 & \text{iff} & \text{not } \left(Pr^{\mathfrak{S}}(s \models \varphi) > p \text{ for all policies } \mathfrak{S} \right)
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Input: a finite MDP $\mathcal{M} = (S, Act, \mathbf{P}, \iota_{\text{init}}, AP, L)$, state $s \in S$, and PCTL state formula Φ

Output: yes, if $s \models \Phi$; no, otherwise.

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3. Check whether state s belongs to $Sat(\Phi)$.

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In order to determine whether $s \in Sat(\mathbb{P}_{\leqslant p}(\varphi))$, the probability $Pr^{\max}(s \models \varphi)$ needs to be established.

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The same holds for strict upper bounds $< p$.

Similarly, lower bounds amount to determining $Pr^{\min}(s \models \varphi)$, e.g.,

$$Sat(\mathbb{P}_{>p}(\varphi)) \;=\; \{s \in S \mid Pr^{\min}(s \models \varphi) > p\}.$$

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Algorithm

Determine $x_s = Pr^{\max}(s \models \bigcirc \Phi)$ and return $Sat(\mathbb{P}_{\leqslant p}(\bigcirc \Phi)) = \{ s \in S \mid x_s \leqslant p \}$.

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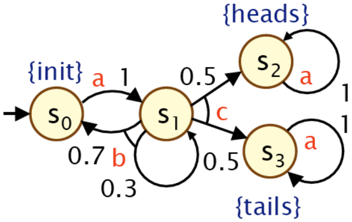
The case for minimal probabilities is similar and omitted here.

Example

Consider MDP:



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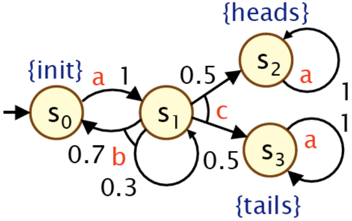

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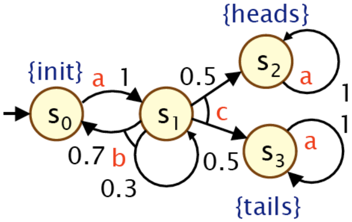

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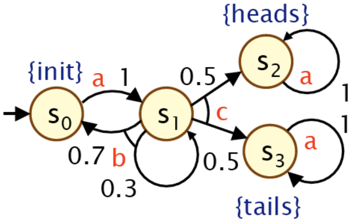

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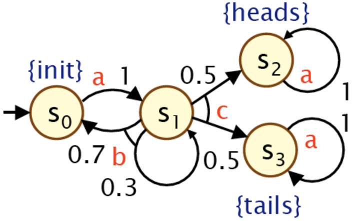

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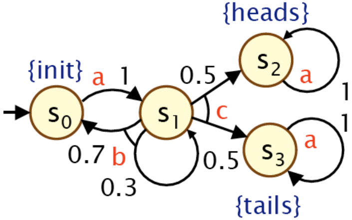

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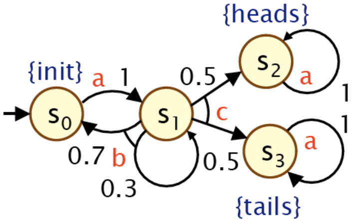

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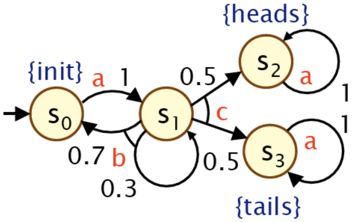

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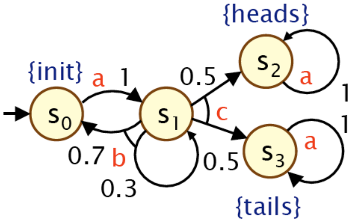

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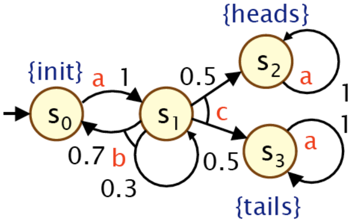

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4. Thus: $Sat(\mathbb{P}_{\geqslant 0.5}(\bigcirc\, heads)) \;=\; \{\, s_2 \,\}$.

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The case for maximal probabilities is analogous.

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1. Let $\mathbf{P}_{\Phi,\Psi}$ be the probability matrix of $\mathcal{M}[S_{=0} \cup S_{=1}]^1$.

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4. This requires n matrix-vector multiplications and n minimum operators.

Example

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Recall that: $s \models \mathbb{P}_{\geqslant p}(\Phi \, \mathsf{U} \, \Psi)$ if and only if $Pr^{\min}(s \models \Phi \, \mathsf{U} \, \Psi) \geqslant p$.

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1. Determine $S_{=1} = Sat(\mathbb{P}_{=1}(\Phi \, \mathrm{U} \, \Psi))$ by a graph analysis.

Until

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4. For qualitative properties, no further computation is needed.

Precomputations

Qualitative reachability

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1. Determine all states for which probability is zero
   1.1 minimum: $\{ s \in S \mid Pr^{\min}(s \models \Phi \, \mathsf{U} \, \Psi) = 0 \}$
   1.2 maximum: $\{ s \in S \mid Pr^{\max}(s \models \Phi \, \mathsf{U} \, \Psi) = 0 \}$
2. Determine all states for which probability is one
   2.1 minimum: $\{ s \in S \mid Pr^{\min}(s \models \Phi \, \mathsf{U} \, \Psi) = 1 \}$
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The first case has been treated in the previous lecture (for $\Diamond G$).

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- Goal is to compute $\{ s \in S \mid Pr^{\max}(s \models \lozenge G) = 1 \}$
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This procedure is quadratic in the size of the MDP.

Algorithm

Algorithm 45 Computing the set of states s with $Pr^{\max}(s \models \Diamond B) = 1$

Input: MDP $\mathcal{M}$ with finite state space S, $B \subseteq S$ for $s \in B : Act(s) = \{ \alpha_s \}$ and $\mathbf{P}(s, \alpha_s, s) = 1$ (i.e., B is absorbing)
Output: $\{ s \in S \mid Pr^{\max}(s \models \Diamond B) = 1 \}$

$U := \{ s \in S \mid s \not\models \exists \Diamond B \}$;
repeat
  $R := U$;
  while $R \neq \varnothing$ **do**
    let $u \in R$;
    $R := R \setminus \{ u \}$;
    for all $(t, \alpha) \in Pre(u)$ such that $t \notin U$ **do**
      remove α from $Act(t)$;
      if $Act(t) = \varnothing$ **then**
        $R := R \cup \{ t \}$;
        $U := U \cup \{ t \}$;
      fi
    od
    (* all incoming edges of u have been removed *)
    remove u and its outgoing edges from $\mathcal{M}$
  od
  (* determine the states s that cannot reach B in the modified MDP *)
  $U := \{s \in S \setminus U \mid s \not\models \exists \Diamond B\}$;
until $U = \varnothing$
(* all states can reach B in the generated sub-MDP of $\mathcal{M}$ *)
return all states in the remaining MDP

Overview

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$$\mathcal{O}\big(\, poly(size(\mathcal{M})) \cdot n_{\max} \cdot |\Phi| \,\big)$$

where $n_{\max} \;=\; \max\{\, n \mid \Psi_1 \, \mathsf{U}^{\leqslant n} \, \Psi_2 \text{ occurs in } \Phi \,\}$ with $\max \varnothing = 1$.

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Is it possible to obtain this information without revealing the identity of the cryptographer that paid?

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Claim

An odd number of agrees indicates that the master paid, while an even number indicates that a cryptographer paid.

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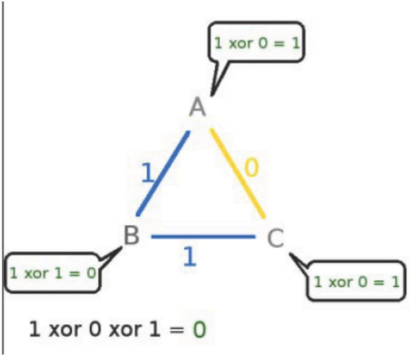

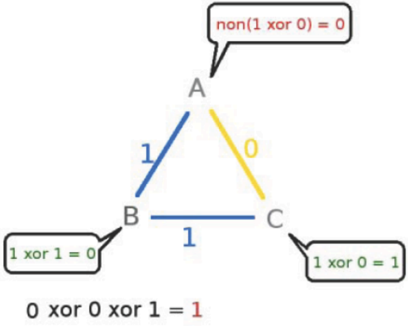

Example scenario in which master paid (left) or cryptographer A paid (right) and provides a misleading vote.

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- if N is even, then an even number of agrees indicates that the master paid while an odd number indicates that a cryptographer paid.

MDP generation times

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N:	Model:		Construction time (s):
	States:	Transitions:	
3	286	585	0.001
4	1,733	4,580	0.01
5	9,876	32,315	0.03
6	54,055	211,566	0.07
7	287,666	1,312,045	0.11
8	1,499,657	7,813,768	0.22
9	7,695,856	45,103,311	0.34
10	39,005,611	253,985,650	0.52
15	115,553,171,626	1,128,594,416,085	3.27
20	304,287,522,253,461	3,962,586,180,540,340	13.48

The number of states and transitions in the MDP representing the model for the dining cryptographers problem with N cryptographers.

Checking correctness

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N:	master pays:		cryptographers pay:	
	time:	iterations:	time:	iterations:
3	0.028	7	0.008	7
4	0.061	9	0.032	9
5	0.141	11	0.085	11
6	0.322	13	0.292	13
7	0.778	15	0.563	15
8	1.467	17	2.25	17
9	2.67	19	4.14	19
10	6.30	21	7.63	21
15	56.9	31	185	31
20	268	41	954	41

$$pay \;\Rightarrow\; \mathbb{P}_{=1}\,(\Diamond(done \wedge par = N\%2)) \;\wedge\; \neg pay \;\Rightarrow\; \mathbb{P}_{=1}\,(\Diamond(done \wedge par \neq N\%2)).$$

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$pay \Rightarrow \mathbb{P}_{=1}(\Diamond(done \wedge par = N\%2)) \wedge \neg pay \Rightarrow \mathbb{P}_{=1}(\Diamond(done \wedge par \neq N\%2))$.
That is: if the master paid, the parity of the number of agrees matches the parity of N and, if a cryptographer paid, it does not.

Checking anonymity

Checking anonymity

N:	minimum:			maximum:		
	time:	iterations:	probability:	time:	iterations:	probability:
3	0.099	8	0.25	0.004	8	0.25
4	0.041	10	0.125	0.006	10	0.125
5	0.172	12	0.0625	0.032	12	0.0625
6	0.231	14	0.03125	0.044	14	0.03125
7	0.595	16	0.015625	0.301	16	0.015625
8	1.111	18	0.0078125	0.540	18	0.0078125
9	2.12	20	0.00390625	1.31	20	0.00390625
10	3.53	22	0.001953125	2.67	22	0.001953125
15	45.1	32	6.103515625E-5	36.8	32	6.103515625E-5

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- Computing minimal reachability probabilities under strongly fair policies is reducible to computing maximal reachability probabilities

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- The until-operator amounts to solving a linear inequation system.

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- Checking whether $s \models \mathbb{P}_{>p}(\varphi)$ amounts to determine $Pr^{\min}(s \models \varphi)$.
- Checking whether $s \models \mathbb{P}_{<p}(\varphi)$ amounts to determine $Pr^{\max}(s \models \varphi)$.
- The next operator amounts to a single matrix-vector multiplication and a max/min.
- The bounded-until operator $\mathsf{U}^{\leqslant n}$ amounts to n matrix-vector multiplications + n minimums (or maximums).
- The until-operator amounts to solving a linear inequation system.
- The worst-case time complexity is polynomial in the size of the MDP and linear in the size of the formula.