Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

November 11, 2015

Overview

Summary of previous lecture

Probabilistic CTL

- Allows for path properties, such as (bounded) until and next.
- State formulas include propositional logic + the operator $\mathbb{P}_J(\varphi)$
- $s \models \mathbb{P}_J(\varphi)$ if the probability of all paths starting in s fulfilling φ is in J
- Model checking is done by a recursive descent over the formula
- This yields a polynomial-time algorithm (linear in $|\Phi|$).

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1. Qualitative PCTL versus CTL.

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2. Qualitative PCTL versus CTL with fairness.

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1. Qualitative PCTL versus CTL.
2. Qualitative PCTL versus CTL with fairness.
3. Repeated reachability probabilities in PCTL.

Overview

PCTL syntax

Probabilistic Computation Tree Logic: Syntax

PCTL consists of state- and path-formulas.

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- PCTL *state formulas* over the set AP obey the grammar:

$$\Phi \ ::= \ \text{true} \ \Big| \ a \ \Big| \ \Phi_1 \wedge \Phi_2 \ \Big| \ \neg\Phi \ \Big| \ \mathbb{P}_J(\varphi)$$

where $a \in AP$, φ is a path formula and $J \subseteq [0, 1]$ is an interval.

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- PCTL *path formulae* are formed according to the following grammar:

$$\varphi \ ::= \ \bigcirc \Phi \ \Big| \ \Phi_1 \, \mathsf{U} \, \Phi_2 \ \Big| \ \Phi_1 \, \mathsf{U}^{\leqslant n} \, \Phi_2$$

  where Φ, Φ_1, and Φ_2 are state formulae and $n \in \mathbb{N}$.

Qualitative PCTL

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State formulae in the *qualitative fragment* of PCTL (over AP):

$$\Phi ::= \text{true} \;\Big|\; a \;\Big|\; \Phi_1 \wedge \Phi_2 \;\Big|\; \neg\Phi \;\Big|\; \mathbb{P}_{>0}(\varphi) \;\Big|\; \mathbb{P}_{=1}(\varphi)$$

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where $a \in AP$, and φ is a path formula formed according to the grammar:

$$\varphi ::= \bigcirc\, \Phi \;\Big|\; \Phi_1 \,\mathsf{U}\, \Phi_2.$$

Remark

The probability bounds $= 0$ and < 1 can be derived:

$$\mathbb{P}_{=0}(\varphi) \equiv \neg\mathbb{P}_{>0}(\varphi) \quad \text{and} \quad \mathbb{P}_{<1}(\varphi) \equiv \neg\mathbb{P}_{=1}(\varphi)$$

So, in qualitative PCTL, there is no bounded until, and only > 0, $= 0$, > 1 and $= 1$ are allowed thresholds.

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$\mathbb{P}_{=1}(\Diamond \mathbb{P}_{>0}(\bigcirc a))$

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Examples

$\mathbb{P}_{=1}(\Diamond \mathbb{P}_{>0}(\bigcirc a))$ and $\mathbb{P}_{<1}(\mathbb{P}_{>0}(\Diamond a) \,\mathsf{U}\, b)$ are qualitative PCTL formulas.

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No bounded until, and only universal and existential path quantifiers.

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Examples

$\forall\Diamond\exists\bigcirc a$ and $\exists(\forall\Diamond a)\,\mathsf{U}\,b$ are CTL formulas.

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Intuition

- $s \models \forall\varphi$ if all paths starting in s fulfill φ

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Question: are CTL and qualitative PCTL equally expressive?

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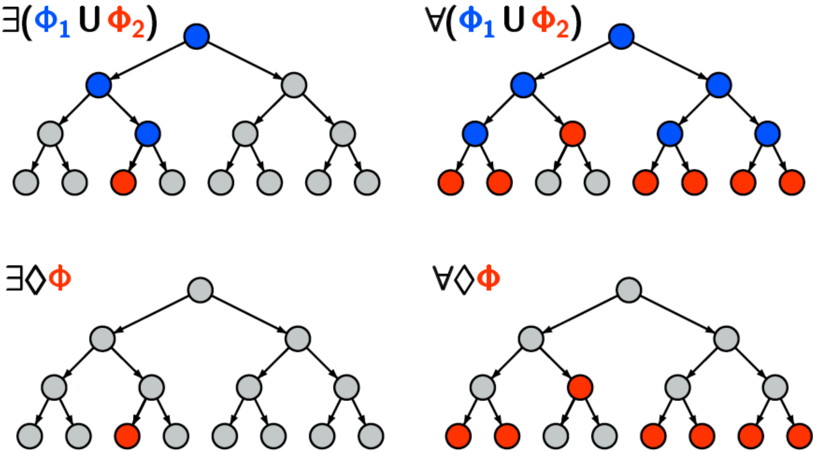
Question: are CTL and qualitative PCTL equally expressive? No.

CTL semantics



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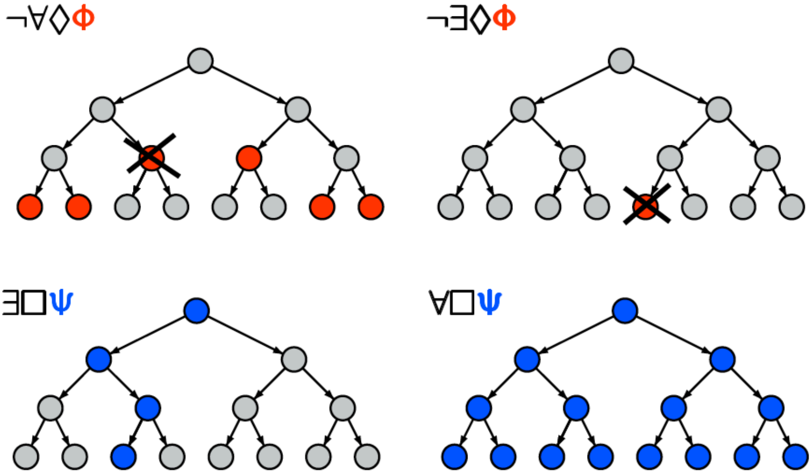

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$\mathcal{D}, s \models \Phi$ if and only if state-formula Φ holds in state s of (possibly infinite) DTMC $\mathcal{D}$. As $\mathcal{D}$ is known from the context we simply write $s \models \Phi$.

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s \models a & \text{iff} & a \in L(s) \\
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s \models \exists\varphi & \text{iff} & \text{there exists } \pi \in Paths(s). \pi \models \varphi \\
s \models \forall\varphi & \text{iff} & \text{for all } \pi \in Paths(s). \pi \models \varphi
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where the semantics of CTL path-formulas is the same as for PCTL path-formulas.

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CTL versus qualitative PCTL

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Equivalence of PCTL and CTL Formulae

The PCTL formula Φ is *equivalent* to the CTL formula Ψ, denoted $\Phi \equiv \Psi$, if $Sat(\Phi) = Sat(\Psi)$ for each DTMC $\mathcal{D}$.

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Example

The simplest such cases are path formulae involving the next-step operator:

$$\begin{aligned} \mathbb{P}_{=1}(\bigcirc a) &\equiv \forall \bigcirc a \\ \mathbb{P}_{>0}(\bigcirc a) &\equiv \exists \bigcirc a \end{aligned}$$

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$\Leftarrow$ Assume $s \models \exists\Diamond a$,

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- $\Leftarrow$ Assume $s \models \exists\Diamond a$, i.e., there is a finite path $\hat{\pi} = s_0\, s_1 \ldots s_n$ with $s_0 = s$ and $s_n \models a$.

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- $\Leftarrow$ Assume $s \models \exists\Diamond a$, i.e., there is a finite path $\hat{\pi} = s_0\, s_1 \ldots s_n$ with $s_0 = s$ and $s_n \models a$. It follows that all paths in the cylinder set $Cyl(\hat{\pi})$ fulfill $\Diamond a$.

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$$Pr(s \models \Diamond a) \;\geqslant\; Pr_s(Cyl(s_0\, s_1 \ldots s_n)) \;=\; \mathbf{P}(s_0 s_1 \ldots s_n) \;>0.$$

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So, by the PCTL semantics we have: $s \models \mathbb{P}_{>0}(\Diamond a)$.

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$\Rightarrow$ Assume $s \models \mathbb{P}_{>0}(\Diamond a)$. By the PCTL semantics, $Pr(s \models \Diamond a) > 0$. Thus, $\{ \pi \in Paths(s) \mid \pi \models \Diamond a \} \neq \varnothing$, and hence, $s \models \exists\Diamond a$.

$\Leftarrow$ Assume $s \models \exists\Diamond a$, i.e., there is a finite path $\hat{\pi} = s_0\, s_1 \ldots s_n$ with $s_0 = s$ and $s_n \models a$. It follows that all paths in the cylinder set $Cyl(\hat{\pi})$ fulfill $\Diamond a$. Thus:

$$Pr(s \models \Diamond a) \ \geqslant \ Pr_s(Cyl(s_0\, s_1 \ldots s_n)) \ = \ \mathbf{P}(s_0 s_1 \ldots s_n) \ > 0.$$

So, by the PCTL semantics we have: $s \models \mathbb{P}_{>0}(\Diamond a)$.

(2) The second statement follows by duality.

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Example

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(1) $\mathbb{P}_{>0}(\Diamond a) \;\equiv\; \exists\Diamond a$ and (2) $\mathbb{P}_{=1}(\Box a) \;\equiv\; \forall\Box a$.

(3) $\mathbb{P}_{>0}(\Box a) \;\not\equiv\; \exists\Box a$ and (4) $\mathbb{P}_{=1}(\Diamond a) \;\not\equiv\; \forall\Diamond a$.

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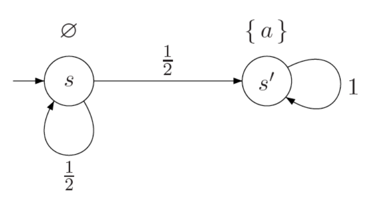

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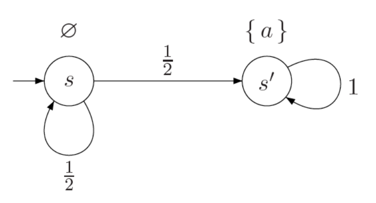

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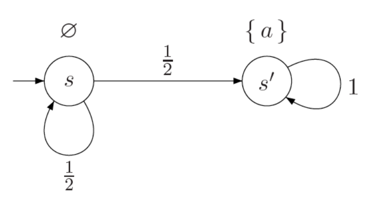

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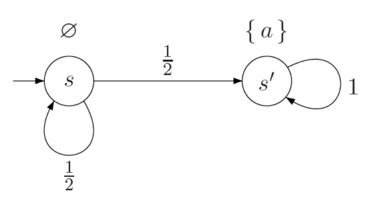

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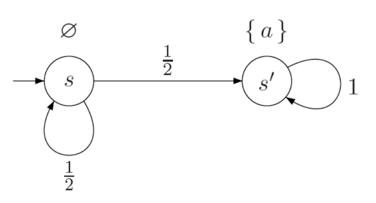

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Statement (3) follows by duality.

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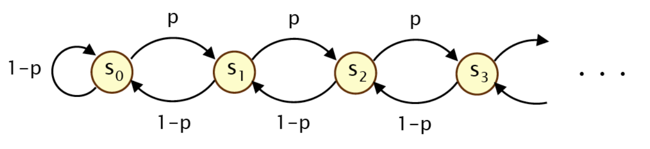

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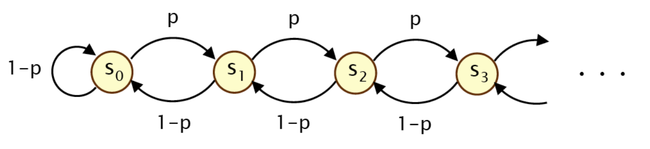

The value of p does affect reachability: $Pr(s \models \Diamond s_0) = \begin{cases} 1 & \text{if } p \leqslant \frac{1}{2} \\ < 1 & \text{if } p > \frac{1}{2} \end{cases}$

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Thus, in $\mathcal{D}_{\frac{1}{4}}$ we have $s \models \mathbb{P}_{=1}(\Diamond s_0)$ for all states s, while in $\mathcal{D}_{\frac{3}{4}}$, e.g., $s_1 \not\models \mathbb{P}_{=1}(\Diamond s_0)$.

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There is no CTL formula that is equivalent to $\mathbb{P}_{=1}(\lozenge a)$.

Proof:

We have: $Pr(s \models \lozenge s_0) = \begin{cases} 1 & \text{if } p \leqslant \frac{1}{2} \\ < 1 & \text{if } p > \frac{1}{2} \end{cases}$

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For each finite DTMC $\mathcal{D}$ it holds that:

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Exercise.

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Proof of the first claim on the black board.

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Qualitative PCTL versus CTL

Incomparable expressiveness

Qualitative PCTL and CTL have incomparable expressiveness; e.g., $\forall\Diamond a$ cannot be expressed in qualitative PCTL and $\mathbb{P}_{=1}(\Diamond a)$ cannot be expressed in CTL.

Overview

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It asserts that when a state s is visited infinitely often, then any of its direct successors is visited infinitely often too.

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$$s \models_{fair} \forall\varphi \quad \text{iff} \quad \text{for all } \pi \in Paths_{fair}(s).\, \pi \models_{fair} \varphi.$$

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Qualitative PCTL versus fair CTL

Comparable expressiveness

Qualitative PCTL and fair CTL are equally expressive for finite Markov chains.

Overview

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On the blackboard.

Remark:

For CTL, universal repeated reachability properties can be formalized by the combination of the modalities $\forall\Box$ and $\forall\Diamond$:

$$s \models \forall\Box\forall\Diamond G \quad \text{iff} \quad \pi \models \Box\Diamond G \text{ for all } \pi \in Paths(s).$$

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By the above theorem, $\mathbb{P}_{>0}(\Box\Diamond G)$ is PCTL definable. Note that $\exists\Box\Diamond G$ is not CTL-definable (but definable in a combination of CTL and LTL, called CTL*).

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Left as an exercise.

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Proof:

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Take-home messages

Qualitative PCTL and CTL have incomparable expressiveness. Qualitative and fair CTL are equally expressive. Repeated reachability and persistence probabilities are PCTL definable. Their qualitative counterparts are not all expressible in CTL.