Modeling and Verification of Probabilistic Systems

Joost-Pieter Katoen

Lehrstuhl für Informatik 2
Software Modeling and Verification Group

http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

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Overview

Summary of previous lecture

Reachability probabilities

Can be obtained as a unique solution of a linear equation system.

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Reachability probabilities are pivotal

The probability of satisfying an ω-regular property P in a Markov chain $\mathcal{D}$ $=$ reachability probability of accepting BSCCs in the product of $\mathcal{D}$ with a DRA for P.

Aim of this lecture

Introduce probabilistic CTL. Provide a polynomial-time model-checking algorithm for verifying a finite Markov chain against a PCTL formula.

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1. Syntax and formal semantics of probabilistic CTL.
2. Model checking algorithm for probabilistic CTL on Markov chains.
3. Time complexity analysis.

Overview

Probabilistic Computation Tree Logic

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- The main operator is $\mathbb{P}_J(\varphi)$
  - where φ constrains the set of paths and J is a threshold on the probability.
  - it is the probabilistic counterpart of $\exists$ and $\forall$ path-quantifiers in CTL.

DTMCs — A transition system perspective

Discrete-time Markov chain

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Initial states

- $\iota_{\mathrm{init}}(s)$ is the probability that DTMC $\mathcal{D}$ starts in state s
- the set $\{ s \in S \mid \iota_{\mathrm{init}}(s) > 0 \}$ are the possible initial states.

PCTL syntax [Hansson & Jonsson, 1994]

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- PCTL *state formulas* over the set AP obey the grammar:

$$\Phi \ ::= \ \text{true} \ \Big| \ a \ \Big| \ \Phi_1 \wedge \Phi_2 \ \Big| \ \neg\Phi \ \Big| \ \mathbb{P}_J(\varphi)$$

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  Abbreviate $\mathbb{P}_{[0,0.5]}(\varphi)$ by $\mathbb{P}_{\leqslant 0.5}(\varphi)$ and $\mathbb{P}_{]0,1]}(\varphi)$ by $\mathbb{P}_{>0}(\varphi)$.

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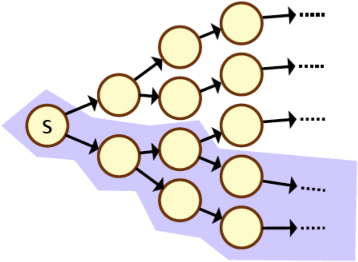

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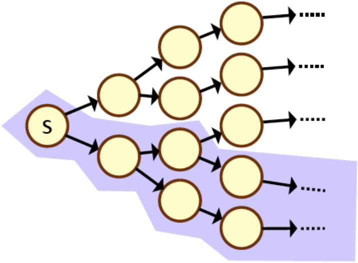

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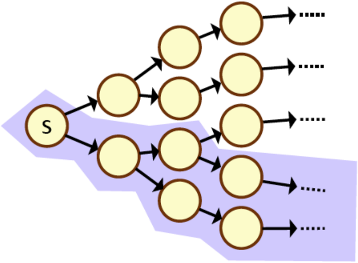

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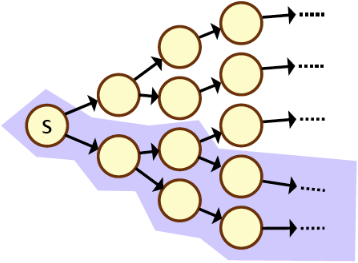

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- Example: $s \models \mathbb{P}_{>\frac{1}{2}}(\Diamond a)$ if
  - the probability to reach an a-labeled state from s exceeds $\frac{1}{2}$.
- Formally:
  - $s \models \mathbb{P}_J(\varphi)$ if and only if $Pr_s\{\, \pi \in Paths(s) \mid \pi \models \varphi \,\} \in J$.

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$$\mathbb{P}_{(p,q)}(\Box^{\leqslant n} \Phi) \;=\; \mathbb{P}_{[1-q,1-p]}(\Diamond^{\leqslant n} \neg \Phi)$$

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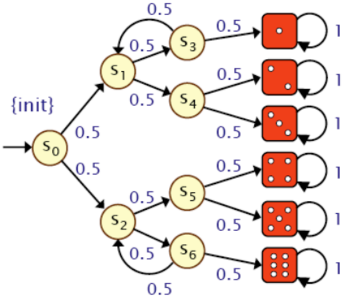

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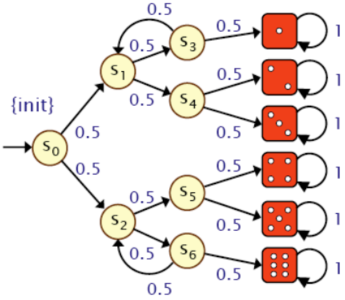

Correctness of Knuth's die

$$\mathbb{P}_{=\frac{1}{6}}(\Diamond 1) \ \wedge\ \mathbb{P}_{=\frac{1}{6}}(\Diamond 2) \ \wedge\ \mathbb{P}_{=\frac{1}{6}}(\Diamond 3) \ \wedge\ \mathbb{P}_{=\frac{1}{6}}(\Diamond 4) \ \wedge\ \mathbb{P}_{=\frac{1}{6}}(\Diamond 5) \ \wedge\ \mathbb{P}_{=\frac{1}{6}}(\Diamond 6)$$

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- . . . in maximally 137 steps: $\mathbb{P}_{\geqslant 0.92}\left(\neg\, illegal\ \mathsf{U}^{\leqslant 137}\ goal\right)$
- . . . once there, remain there almost surely for the next 31 steps:

$$\mathbb{P}_{\geqslant 0.92}\left(\neg\, illegal\ \mathsf{U}^{\leqslant 137}\ \mathbb{P}_{=1}(\Box^{[0,31]}\ goal)\right)$$

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Notation

$\mathcal{D}, s \models \Phi$ if and only if state-formula Φ holds in state s of (possibly infinite) DTMC $\mathcal{D}$. As $\mathcal{D}$ is known from the context we simply write $s \models \Phi$.

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Satisfaction relation for state formulas

The satisfaction relation $\models$ is defined for PCTL state formulas by:

$$\begin{array}{lll} s \models a & \text{iff} & a \in L(s) \\ s \models \neg \Phi & \text{iff} & \text{not } (s \models \Phi) \\ s \models \Phi \wedge \Psi & \text{iff} & (s \models \Phi) \text{ and } (s \models \Psi) \end{array}$$

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s \models \mathbb{P}_J(\varphi) & \text{iff} & Pr(s \models \varphi) \in J
\end{array}
$$

where $Pr(s \models \varphi) = Pr_s\{\,\pi \in Paths(s) \mid \pi \models \varphi\,\}$

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\pi \models \Phi \, \mathsf{U}^{\leqslant n} \, \Psi & \text{iff} & \exists k \geqslant 0. (\, k \leqslant n \text{ and } \pi[k] \models \Psi \text{ and} \\
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Examples

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Proof (sketch):

Three cases:

1. $\bigcirc \Phi$:
   - cylinder sets constructed from paths of length one.
2. $\Phi \, \mathsf{U}^{\leqslant n} \, \Psi$:
   - (finite number of) cylinder sets from paths of length at most n.
3. $\Phi \, \mathsf{U} \, \Psi$:
   - countable union of paths satisfying $\Phi \, \mathsf{U}^{\leqslant n} \, \Psi$ for all $n \geqslant 0$.

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PCTL model checking problem

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   - Determine $Sat(\Psi)$ as function of the satisfaction sets of its children:

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3. Check whether state s belongs to $Sat(\Phi)$.

Example

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Propositional formulas

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$Sat(\cdot)$ is defined by structural induction as follows:

$$
\begin{aligned}
Sat(\mathrm{true}) &= S \\
Sat(a) &= \{\, s \in S \mid a \in L(s) \,\}, \text{ for any } a \in AP \\
Sat(\Phi \wedge \Psi) &= Sat(\Phi) \cap Sat(\Psi) \\
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Let us consider the computation of $Pr(s \models \varphi)$ for all possible φ.

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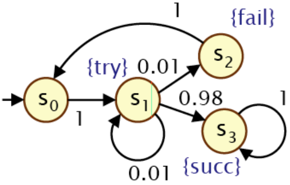

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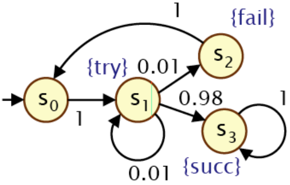

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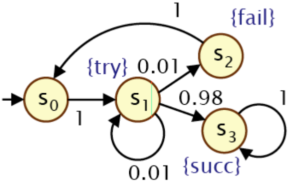

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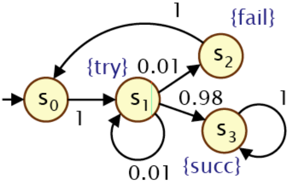

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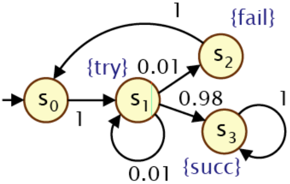

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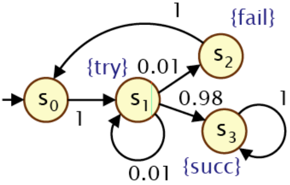

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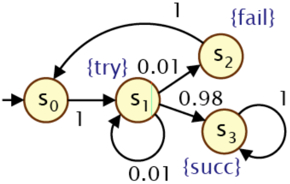

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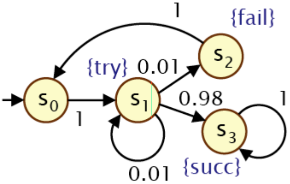

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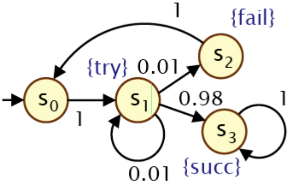

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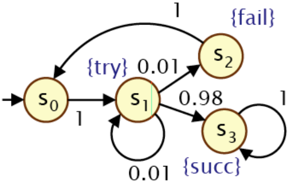

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4. Thus: $Sat(\mathbb{P}_{\geqslant 0.9}(\bigcirc\,(\neg try \vee succ)) = \{ s_1, s_2, s_3 \}$.

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3. And $\left(Pr(s \models \Phi\, \mathsf{U}^{\leqslant i+1}\, \Psi)\right)_{s \in S} \;=\; \mathbf{P}_{\Phi,\Psi} \cdot \left(Pr(s \models \Phi\, \mathsf{U}^{\leqslant i}\, \Psi)\right)_{s \in S}$.
4. This requires n matrix-vector multiplications in total.

Remarks

1. In terms of matrix powers: $\left(Pr(s \models \Phi\, \mathsf{U}^{\leqslant n}\, \Psi)\right)_{s \in S} \;=\; \mathbf{P}^{n}_{\Phi,\Psi} \cdot \mathbf{b}_{\Psi}$.
   - Computing $\mathbf{P}^{n}_{\Phi,\Psi}$ in $\log_2 n$ steps is inefficient due to fill-in.
   - That is to say, $\mathbf{P}^{n}_{\Phi,\Psi}$ is much less sparse than $\mathbf{P}_{\Phi,\Psi}$.
2. $\mathbf{P}^{n}_{\Phi,\Psi} \cdot \mathbf{b}_{\Psi} \;=\; \left(Pr(s \models \bigcirc^{=n}\, \Psi)\right)_{s \in S_?}$ in $\mathcal{D}[S_{=0} \cup S_{=1}]$.
   - Where $\bigcirc^{0}\, \Psi = \Psi$ and $\bigcirc^{i+1}\, \Psi = \bigcirc\, (\bigcirc^{i}\, \Psi)$.
   - This thus amounts to a transient analysis in DTMC $\mathcal{D}[S_{=0} \cup S_{=1}]$.

Optimization

The above procedure used:

- $S_{=1} = Sat(\Psi)$, and
- $S_{=0} = S \setminus (Sat(\Phi) \cup Sat(\Psi)) = Sat(\neg\Phi \wedge \neg\Psi)$, and
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This can be optimized (in practice) by enlarging $S_{=0}$ and$S_{=1}$:

- $S_{=1} = Sat(\mathbb{P}_{=1}(\Phi \,\mathsf{U}\, \Psi))$, obtained by a graph analysis
- $S_{=0} = Sat(\mathbb{P}_{=0}(\Phi \,\mathsf{U}\, \Psi))$, obtained by a graph analysis too, and
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Example

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4. For qualitative properties, no further computation is needed.

Example

Overview

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where $n_{\max} = \max\{ n \mid \Psi_1 \, \mathsf{U}^{\leqslant n} \Psi_2 \text{ occurs in } \Phi \}$ with and $n_{\max} = 1$ if Φ does not contain a bounded until-operator.

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3. Strictly speaking, $\mathrm{U}^{\leqslant n}$ could be more expensive for large n.

   But it remains polynomial, and n is small in practice.

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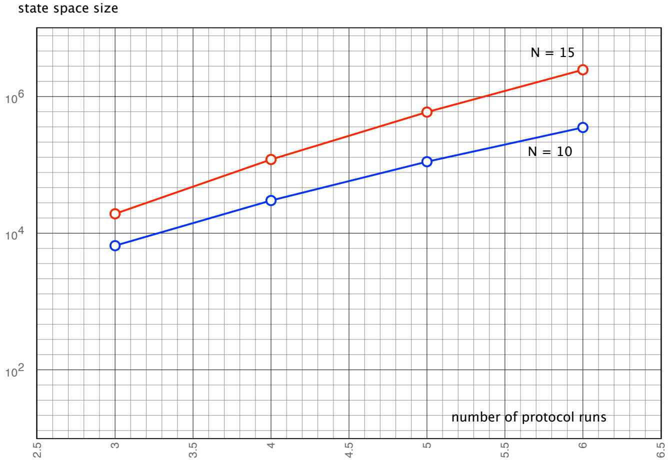
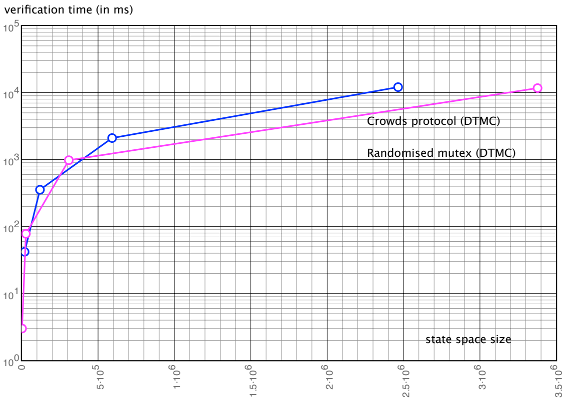
- Rebuild routing paths on crowd changes
- Property: Crowds protocol ensures "probable innocence":
  - probability real sender is discovered $< \frac{1}{2}$ if $N \geqslant \frac{p}{p-\frac{1}{2}} \cdot (c+1)$
  - where N is crowd's size and c is number of corrupt crowd members

State space growth



Some practical verification times



- command-line tool MRMC ran on a Pentium 4, 2.66 GHz, 1 GB RAM laptop.
- PCTL formula $\mathbb{P}_{\leqslant p}(\Diamond obs)$ where *obs* holds when the sender's id is detected.

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- The bounded-until operator $\mathsf{U}^{\leqslant n}$ amounts to n matrix-vector multiplications.
- The until-operator amounts to solving a linear equation system.
- Worst-case time complexity of $\mathcal{D} \models \Phi$ is polynomial in $|\mathcal{D}|$ and linear in $|\Phi|$.