Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

November 4, 2015

Overview

Summary of previous lectures

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Reachability probabilities are pivotal

1. Repeated reachability
   - = Reachability of the BSCCs containing a goal state
2. Persistence
   - = Reachability of the BSCCs only containing goal states

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Intuition

An LT-property gives the admissible behaviours of the DTMC at hand.

Probability of LT properties

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The *probability* for DTMC $\mathcal{D}$ to exhibit a trace in property P (over AP) is:

$$Pr^{\mathcal{D}}(P) \; = \; Pr^{\mathcal{D}}\{\, \pi \in Paths(\mathcal{D}) \mid trace(\pi) \in P \,\}.$$

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We do not address measurability of P yet. We will later identify a rich set P of LT-properties—those that include all LTL formulas—for which the set of paths $\{\, \pi \in Paths(\mathcal{D}) \mid trace(\pi) \in P \,\}$ is measurable.

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LT property P_{safe} over AP is a *safety property* if for all $\sigma \in (2^{AP})^\omega \setminus P_{safe}$ there exists a finite prefix $\widehat{\sigma}$ of σ such that:

$$P_{safe} \cap \underbrace{\left\{\sigma' \in (2^{AP})^\omega \mid \widehat{\sigma} \text{ is a prefix of } \sigma'\right\}}_{\text{all possible extensions of } \widehat{\sigma}} = \varnothing.$$

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Property of Knuth's die

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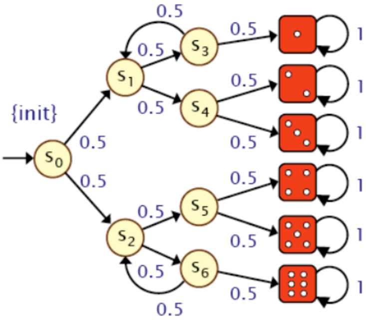

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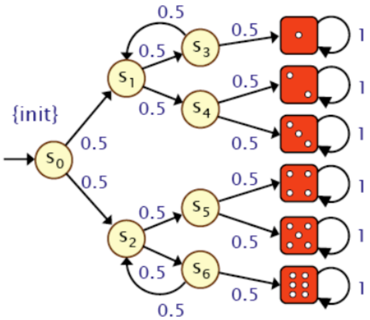

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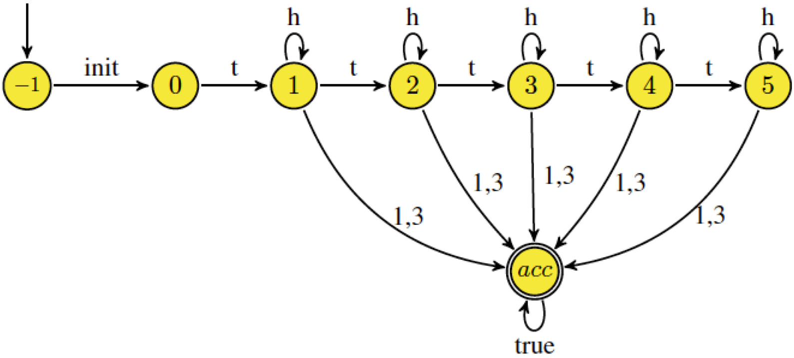
After initial tails, yield 1 or 3 but with maximally five time tails.

Property as an automaton



After initial tails, yield 1 or 3 but with at most five times tails in total

Overview

Probability of a regular safety property

Let $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, F)$ be a deterministic finite-state automaton (DFA) for the bad prefixes of regular safety property P_{safe}:

$$P_{safe} \;=\; \{\, A_0\, A_1\, A_2 \ldots \in \left(2^{AP}\right)^{\omega} \mid \forall n \geqslant 0.\, A_0\, A_1 \ldots A_n \notin \mathcal{L}(\mathcal{A}) \,\}.$$

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These probabilities can be obtained by considering a product of DTMC $\mathcal{D}$ with DFA $\mathcal{A}$.

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The value $Pr(s \models \mathcal{A})$ can be written as the (possibly infinite) sum:

$$Pr(s \models \mathcal{A}) \;=\; \sum_{\widehat{\pi}} \mathbf{P}(\widehat{\pi})$$

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2. the length of $\widehat{\pi}$ is minimal, i.e., $trace(s_0\, s_1 \ldots s_i) \notin \mathcal{L}(\mathcal{A})$ for all $0 \leqslant i < n$.

Product construction: intuition

DTMC $\mathcal{D}$
with state space S

DRA $\mathcal{A}$
with state space Q

$$
\begin{array}{ccc}
s_0 & \qquad L(s_0)=A_0 \\
\downarrow & \\
s_1 & \qquad L(s_1)=A_1 \\
\downarrow & \\
s_2 & \qquad L(s_2)=A_2 \\
\downarrow & \\
\vdots & \qquad \vdots \\
\downarrow & \\
s_n & \qquad L(s_n)=A_n
\end{array}
$$

path

$$
\begin{array}{c}
q_0 \in Q_0 \\
\downarrow A_0 \\
q_1 \\
\downarrow A_1 \\
q_2 \\
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\vdots \\
\downarrow \\
q_n \\
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q_{n+1}
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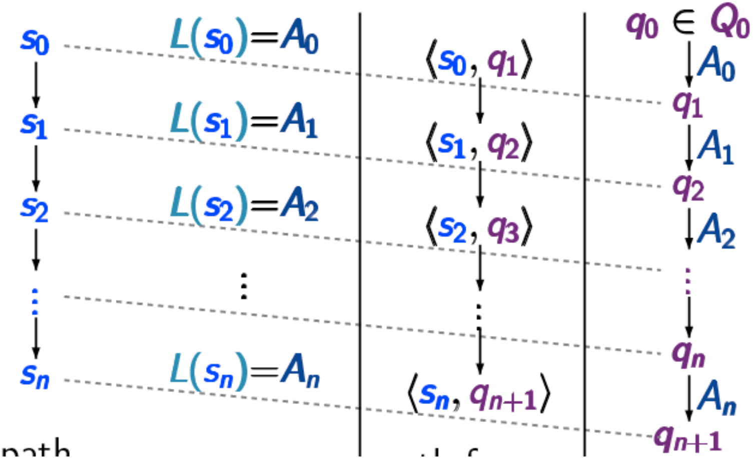

product $\mathcal{D} \otimes \mathcal{A}$

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$$\iota'_{\text{init}}(\langle s, q \rangle) \;=\; \begin{cases} \iota_{\text{init}}(s) & \text{if } q = \delta(q_0, L(s)) \\ 0 & \text{otherwise.} \end{cases}$$

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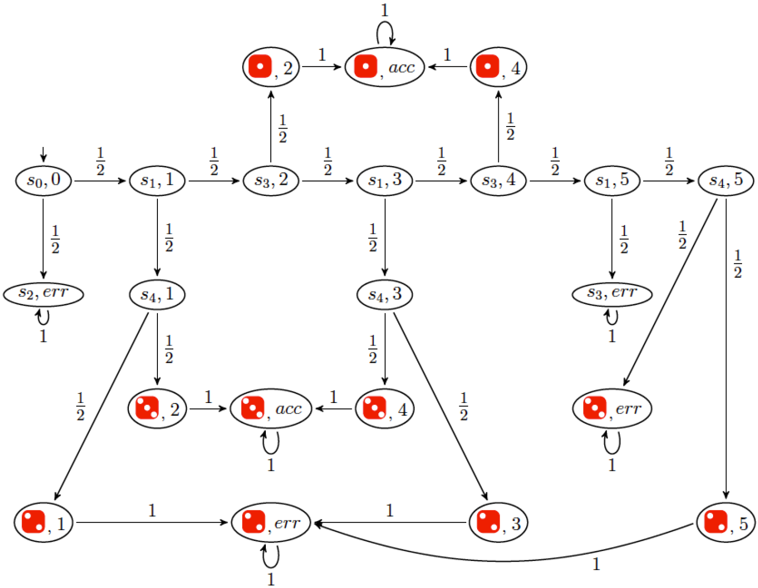
The transition probabilities in $\mathcal{D} \otimes \mathcal{A}$ are given by:

$$\mathbf{P}'(\langle s, q \rangle, \langle s', q' \rangle) \;=\; \begin{cases} \mathbf{P}(s, s') & \text{if } q' = \delta(q, L(s')) \\ 0 & \text{otherwise.} \end{cases}$$

Example product: Knuth-Yao's die



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Some observations

- For each path $\pi = s_0\, s_1\, s_2 \ldots$ in DTMC $\mathcal{D}$ there exists a unique run $q_0\, q_1\, q_2 \ldots$ in DFA $\mathcal{A}$ for $trace(\pi) = L(s_0)\, L(s_1)\, L(s_2) \ldots$ and $\pi^+ = \langle s_0, q_1 \rangle \langle s_1, q_2 \rangle \langle s_2, q_3 \rangle \ldots$ is a path in $\mathcal{D} \otimes \mathcal{A}$.

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- The DFA $\mathcal{A}$ does not affect the probabilities, i.e., for each measurable set Π of paths in $\mathcal{D}$ and state s:

$$Pr_s^{\mathcal{D}}(\Pi) \;=\; Pr^{\mathcal{D}\otimes\mathcal{A}}_{\langle s, \delta(q_0, L(s)) \rangle} \underbrace{\{ \pi^+ \mid \pi \in \Pi \}}_{\Pi^+}$$

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- For $\Pi = \{ \pi \in \mathit{Paths}^{\mathcal{D}}(s) \mid \mathit{pref}(\mathit{trace}(\pi)) \cap \mathcal{L}(\mathcal{A}) \neq \varnothing \}$, the set Π^+ is given by:

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Quantitative analysis of regular safety properties

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Let P_{safe} be a regular safety property, $\mathcal{A}$ a DFA for the set of bad prefixes of P_{safe}, $\mathcal{D}$ a DTMC, and s a state in $\mathcal{D}$. Then:

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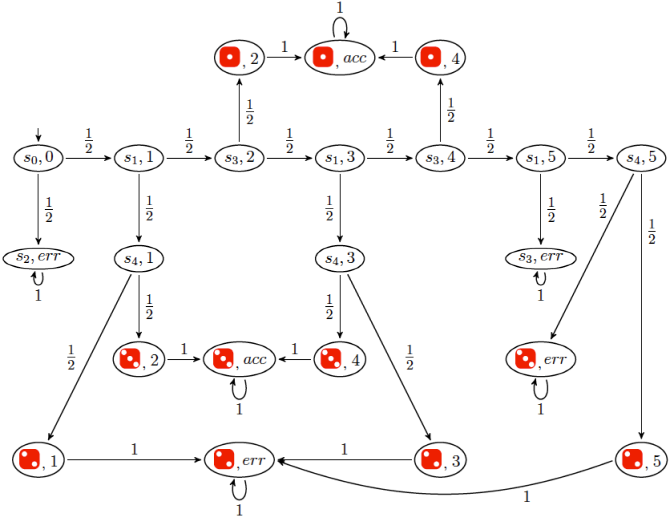
2. For qualitative regular safety properties, i.e., $Pr^{\mathcal{D}}(s \models P_{safe}) > 0$ and $Pr^{\mathcal{D}}(s \models P_{safe}) = 1$, a graph analysis of $\mathcal{D} \otimes \mathcal{A}$ suffices.

Determining the property's probability



$Pr^{\mathcal{D} \otimes \mathcal{A}}(\langle s, q_s \rangle \models \Diamond accept)$ equals $\frac{1}{8} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} = \frac{5}{16}$.

Overview

ω-regular languages

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Infinite repetition of languages

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Example

Examples for ω-regular expressions over the alphabet $\Sigma = \{ A, B, C \}$ are

$$(A + B)^* A(AAB + C)^\omega \quad \text{or} \quad A(B + C)^* A^\omega + B(A + C)^\omega.$$

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- infinitely often a, i.e., $((\varnothing + \{ b \})^*.(\{ a \} + \{ a, b \}))^\omega$.
- from some moment on, always a, i.e., $(2^{AP})^*.(\{ a \} + \{ a, b \})^\omega$.

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$$\left(2^{AP}\right)^\omega \setminus P_{safe} \;=\; \underbrace{BadPref(P_{safe})}_{\text{regular}} . \left(2^{AP}\right)^\omega$$

is an ω-regular language, and ω-regular languages are closed under complement.

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$$\big((\neg \mathit{wait})^*.\mathit{wait}.\mathrm{true}^*.\mathit{crit}\big)^\omega \; + \; \big((\neg \mathit{wait})^*.\mathit{wait}.\mathrm{true}^*.\mathit{crit}\big)^*.(\neg \mathit{wait})^\omega$$

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Intuitively, the first summand stands for the case where $\mathcal{P}$ requests and enters its critical section infinitely often, while the second summand stands for the case where $\mathcal{P}$ is in its waiting phase only finitely many times.

Overview

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A *run* for $\sigma = A_0 A_1 A_2 \ldots \in \Sigma^\omega$ denotes an infinite sequence $q_0\, q_1\, q_2 \ldots$ of states in $\mathcal{A}$ such that $q_0 \in Q_0$ and $q_i \xrightarrow{A_i} q_{i+1}$ for $i \geqslant 0$.

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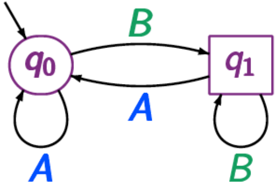
The infinite *language* of $\mathcal{A}$ is

$$\mathcal{L}_\omega(\mathcal{A}) \ = \ \{\, \sigma \in \Sigma^\omega \mid \text{there exists an accepting run for } \sigma \text{ in } \mathcal{A} \,\}.$$

Deterministic Büchi automata for LT properties



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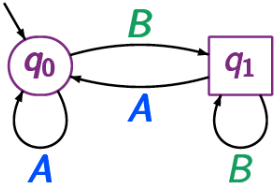

DBA over $\{ A, B \}$ with $F = \{ q_1 \}$ and initial state q_0 accepting the LT property "infinitely often B".

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An ω-language is recognizable by a DBA iff it is the limit language of a regular language. (Details: see lecture Applications of Automata Theory.)

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Quantitative Analysis for DBA-Definable Properties

Let $\mathcal{A}$ be a DBA and $\mathcal{D}$ a DTMC. Then, for all states s in $\mathcal{D}$:

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Quantitative Analysis for DBA-Definable Properties

Let $\mathcal{A}$ be a DBA and $\mathcal{D}$ a DTMC. Then, for all states s in $\mathcal{D}$:

$$Pr^{\mathcal{D}}(s \models \mathcal{A}) \;=\; Pr^{\mathcal{D}\otimes\mathcal{A}}(\langle s, q_s \rangle \models \Box\Diamond accept)$$

where $q_s = \delta(q_0, L(s))$.

Algorithm

Recall that for finite DTMCs, the probability of $\Box\Diamond accept$ can be obtained in polynomial time by first determining the BSCCs of $\mathcal{D} \otimes \mathcal{A}$. For each BSCC B that contains a state $\langle s, q \rangle$ with $q \in F$, determine the probability of eventually reaching B. Its sum is the required probability. Thus this amounts to solve a linear equation system for each accepting BSCC in $\mathcal{D}$.

Overview

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- To overcome this deficiency, Büchi automata will be replaced by an alternative automaton model for which their deterministic counterparts are as expressive as ω-regular languages.
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- To overcome this deficiency, Büchi automata will be replaced by an alternative automaton model for which their deterministic counterparts are as expressive as ω-regular languages.
- Such automata have the same components as DBA (finite set of states, and so on) except for the acceptance sets. We consider *deterministic Rabin automata*. There are alternatives, e.g., Muller automata.
- Determinism is important to stay within the realm of Markov chains; a product of an MC with a deterministic automaton yields a MC.

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Run $q_0\, q_1\, q_2 \ldots$ is *accepting* if for some pair (L_i, K_i), the states in L_i are visited finitely often and the states in K_i infinitely often. That is, an accepting run should satisfy

$$\bigvee_{0<i\leqslant k} (\Diamond\Box\neg L_i \wedge \Box\Diamond K_i).$$

When does a DRA accept an infinite word?

Acceptance condition

A run of a word in Σ^ω on a DRA is accepting if and only if:

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A deterministic Büchi automaton is a DRA with acceptance condition $\{ (\varnothing, F) \}$.

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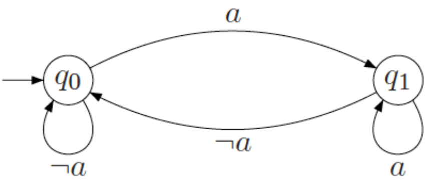

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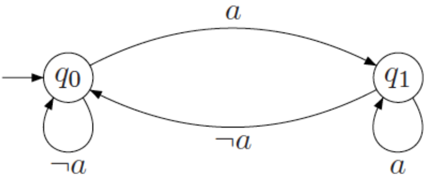

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Recall that there does not exist a deterministic Büchi automaton for $\Diamond \Box a$.

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- DRA are thus equally expressive as nondeterministic Büchi automata.
- They are more expressive than deterministic Büchi automata.
- Any nondeterministic Büchi automata of n states can be converted to a DRA of size $2^{\mathcal{O}(n \cdot \log n)}$. (Details omitted.)

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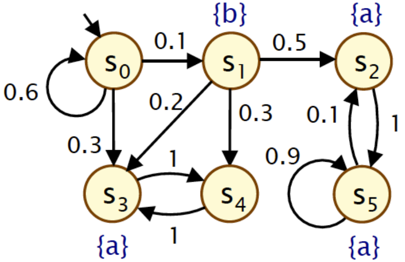
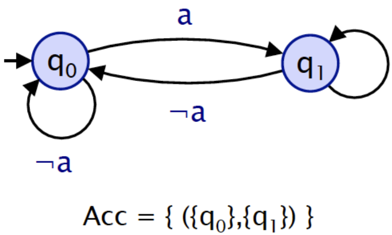
Probabilities for satisfying ω-regular properties are obtained by computing the reachability probabilities for accepting BSCCs in $\mathcal{D} \otimes \mathcal{A}$. Again, a graph analysis and solving systems of linear equations suffice. The time complexity is polynomial in the size of $\mathcal{D}$ and $\mathcal{A}$.

Example: verifying a DTMC versus a DRA





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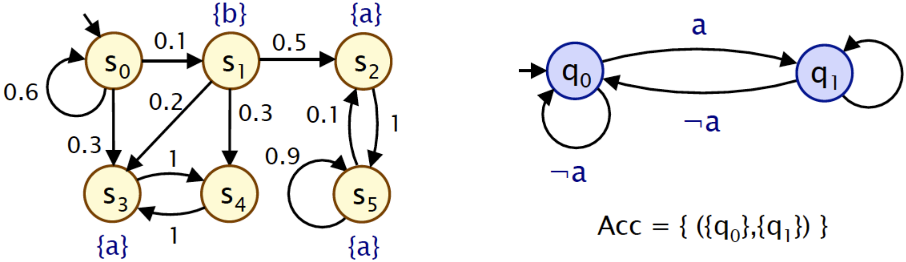

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Reachability probability is $\frac{1}{2} \cdot \frac{1}{10} \cdot \sum_{k=0}^{\infty} \left(\frac{3}{5} \right)^k \; = \; \frac{1}{8}$.

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Measurability theorem for ω-regular properties **[Vardi 1985]**

For any DTMC $\mathcal{D}$ and DRA $\mathcal{A}$ the set

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Let DRA $\mathcal{A}$ with accept sets $\{ (L_1, K_1), \ldots, (L_m, K_m) \}$. Let $\varphi_i = \Diamond \Box \neg L_i \wedge \Box \Diamond K_i$ and Π_i the set of paths satisfying φ_i. Then $\Pi = \Pi_1 \cup \ldots \cup \Pi_k$. In addition, $\Pi_i = \Pi_i^{\Diamond\Box} \cap \Pi_i^{\Box\Diamond}$ where $\Pi_i^{\Diamond\Box}$ is the set of paths π in $\mathcal{D}$ such that $\pi^+ \models \Diamond\Box\neg L_i$, and $\Pi_i^{\Box\Diamond}$ is the set of paths π in $\mathcal{D}$ such that $\pi^+ \models \Box\Diamond K_i$. It remains to show that $\Pi_i^{\Diamond\Box}$ and $\Pi_i^{\Box\Diamond}$ are measurable.

Measurability

Measurability theorem for ω-regular properties **[Vardi 1985]**

For any DTMC $\mathcal{D}$ and DRA $\mathcal{A}$ the set

$$\{\, \pi \in Paths(\mathcal{D}) \mid trace(\pi) \in \mathcal{L}_\omega(\mathcal{A}) \,\}$$

is measurable.

Proof (sketch)

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Linear temporal logic

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Linear Temporal Logic: Syntax [Pnueli 1977]

LTL formulas over the set AP obey the grammar:

$$\varphi \ ::= \ a \ \Big| \ \neg\varphi \ \Big| \ \varphi_1 \wedge \varphi_2 \ \Big| \ \bigcirc \varphi \ \Big| \ \varphi_1 \, \mathsf{U} \, \varphi_2$$

where $a \in AP$ and φ, φ_1, and φ_2 are LTL formulas.

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Example

On the blackboard.

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$$\begin{array}{lcll}
\sigma & \models & \text{true} & \\
\sigma & \models & a & \text{iff} \quad a \in A_0 \quad (\text{i.e., } A_0 \models a) \\
\sigma & \models & \varphi_1 \wedge \varphi_2 & \text{iff} \quad \sigma \models \varphi_1 \text{ and } \sigma \models \varphi_2 \\
\sigma & \models & \neg\varphi & \text{iff} \quad \sigma \not\models \varphi \\
\sigma & \models & \bigcirc\varphi & \text{iff} \quad \sigma^1 = A_1 A_2 A_3 \ldots \models \varphi \\
\sigma & \models & \varphi_1 \, \mathsf{U} \, \varphi_2 & \text{iff} \quad \exists j \geqslant 0.\ \sigma^j \models \varphi_2 \text{ and } \sigma^i \models \varphi_1,\ 0 \leqslant i < j
\end{array}$$

for $\sigma = A_0 A_1 A_2 \ldots$ we have $\sigma^i = A_i A_{i+1} A_{i+2} \ldots$ is the suffix of σ from index i on.

Some facts about LTL

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For any LTL formula φ, there exists a DRA $\mathcal{A}$ such that $\mathcal{L}_\omega = \mathit{Words}(\varphi)$ where the number of states in $\mathcal{A}$ lies in $2^{2^{|\varphi|}}$.

Verifying a DTMC against LTL formulas

Complexity of LTL model checking **[Vardi 1985]**

The qualitative model-checking problem for finite DTMCs against LTL formula φ is PSPACE-complete, i.e., verifying whether $Pr(s \models \varphi) > 0$ or $Pr(s \models \varphi) = 1$ is PSPACE-complete.

Recall that the LTL model-checking problem for finite transition systems is PSPACE-complete.

Overview

1. Introduction
2. Preliminaries
3. Verifying regular safety properties
4. ω-regular properties
5. Verifying DBA objectives
6. Verifying ω-regular properties
7. **Summary**

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Take-home message

Model checking a DTMC against various automata models reduces to computing reachability probabilities in a product.