# Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

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# Overview

# Summary of previous lecture

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- These transient probabilities satisfy: $\Theta_n^{\mathcal{D}} = \iota_{\text{init}} \cdot \mathbf{P}^n$.

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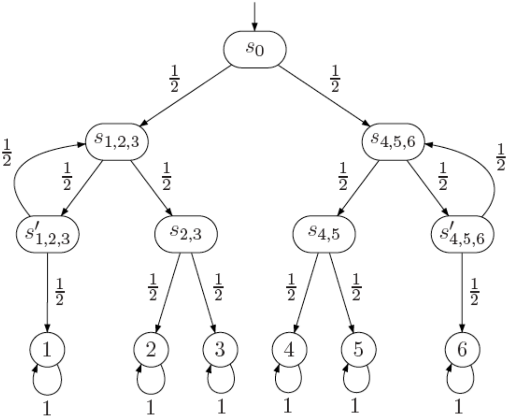
# Recall Knuth's die

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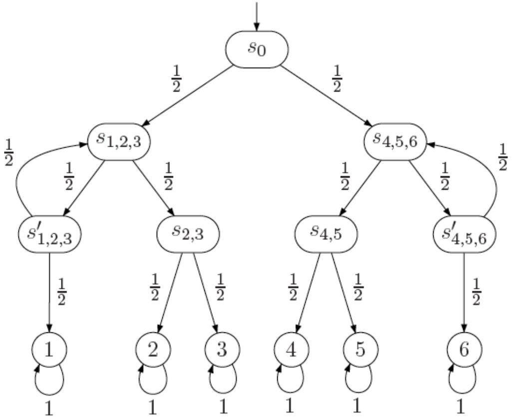


Heads = "go left"; tails = "go right". Does this DTMC model a six-sided die?

# Overview

# Paths

## State graph

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Let $Pre(s)$ be the *predecessors* of $s$, $Pre^*(s)$ its reflexive and transitive closure.

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*Paths* in $\mathcal{D}$ are infinite paths in its state graph.

$Paths(\mathcal{D})$ denotes the set of paths in $\mathcal{D}$, and $Paths^*(\mathcal{D})$ its finite prefixes.

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## Repeated reachability

Repeatedly visit a state in $G$; formally:

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## Persistence

Eventually reach in a state in $G$ and always stay there; formally:

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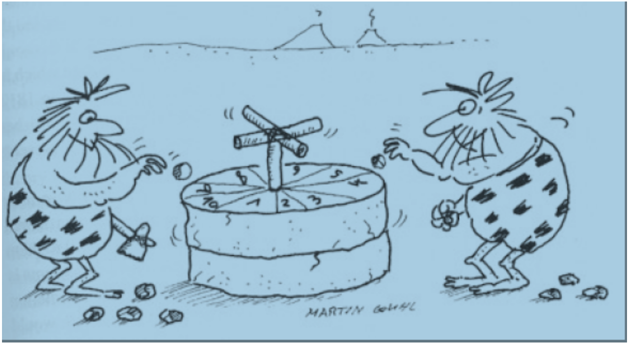
Let $\Omega$ be a set. $\mathcal{F} = \{ \varnothing, \Omega \}$ yields the smallest $\sigma$-algebra; $\mathcal{F} = 2^{\Omega}$ yields the largest one.

# What's the probability of infinite paths?



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  2. $Pr\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} Pr(A_i)$ for any $A_i \in \mathcal{F}$ with $A_i \cap A_j = \varnothing$ for $i \neq j$

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The events in $\mathcal{F}$ of a probability space $(\Omega, \mathcal{F}, Pr)$ are called *measurable*.

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- These events are defined using cylinder sets.
- Cylinder set of a finite path := set of all its infinite continuations.

# Probability measure on DTMCs

## Cylinder set

The *cylinder set* of finite path $\hat{\pi} = s_0\, s_1 \ldots s_n \in Paths^*(\mathcal{D})$ is defined by:

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## Probability space of a DTMC

The set of events of the probability space DTMC $\mathcal{D}$ contains all cylinder sets $Cyl(\hat{\pi})$ where $\hat{\pi}$ ranges over all finite paths in $\mathcal{D}$.

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Thus $\Diamond G$ is measurable.

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$$\lozenge G \;=\; \bigcup_{s_0 \ldots s_n \in Paths^*(\mathcal{D}) \cap (S \setminus G)^* G} Cyl(s_0 \ldots s_n)$$

Thus $\lozenge G$ is measurable.

As all cylinder sets are pairwise disjoint, its probability is defined by:

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Which event does $\Diamond G$ exactly mean?

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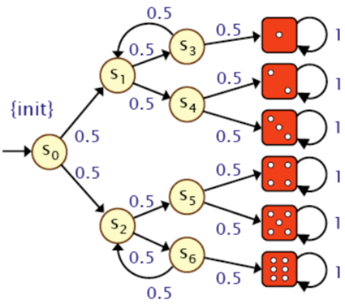
A similar proof strategy applies to the case $\overline{F} \,\mathsf{U}\, G$.

# Proof for $\Box\Diamond G$

# Reachability probabilities: Knuth's die

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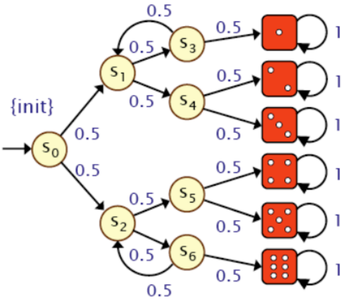


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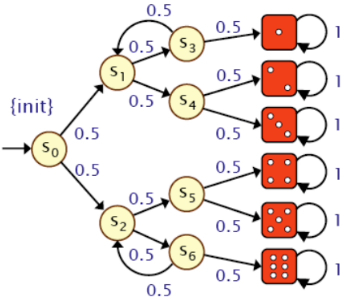


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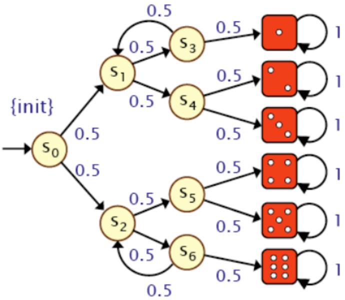


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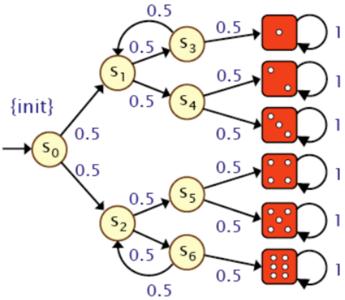


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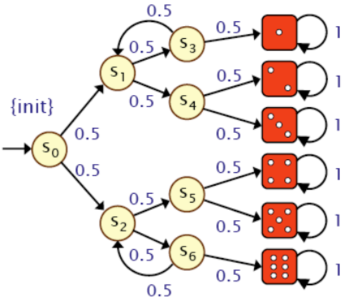


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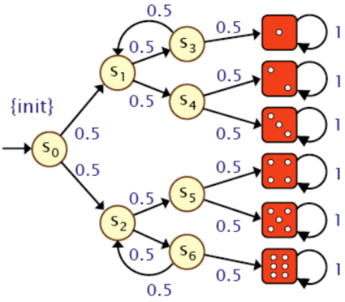


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There is however an simpler way to obtain reachability probabilities!

# Overview

# Reachability probabilities in finite DTMCs

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Let $\mathcal{D}$ be a DTMC with finite state space $S$, $s \in S$ and $G \subseteq S$.

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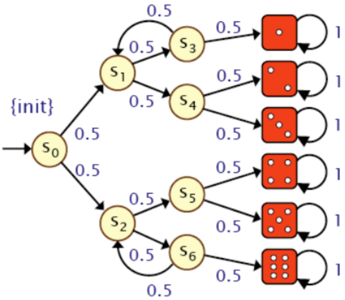


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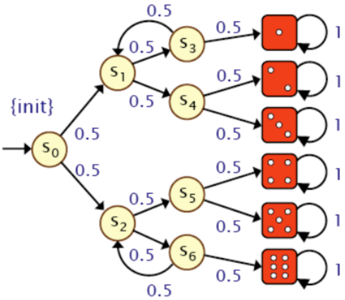


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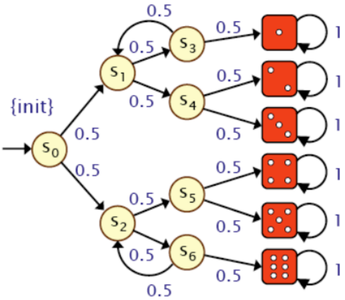


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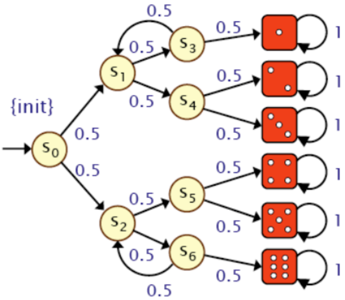


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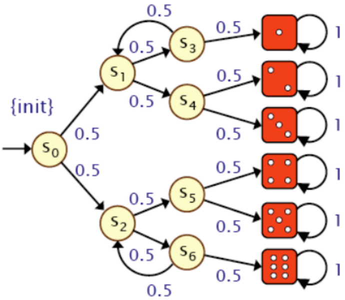


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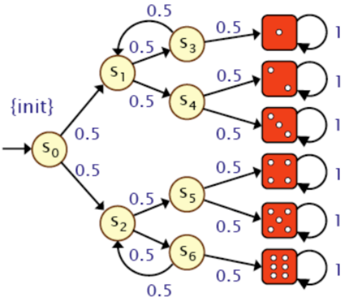


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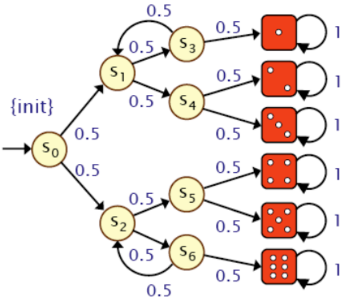


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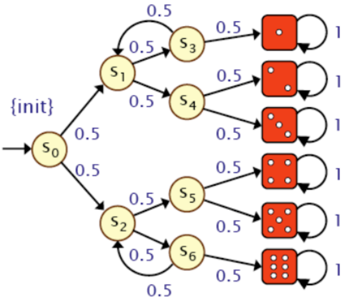


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$$\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b} \quad \text{or} \quad (\mathbf{I} - \mathbf{A}) \cdot \mathbf{x} = \mathbf{b}$$

where $\mathbf{I}$ is the identity matrix of cardinality $|S_?| \times |S_?|$.

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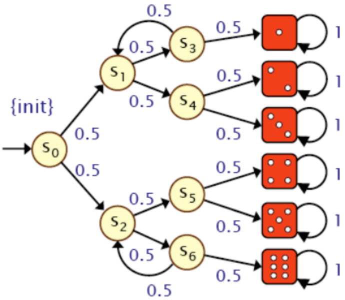


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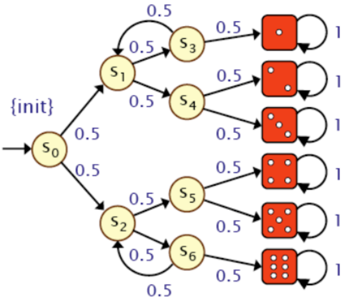


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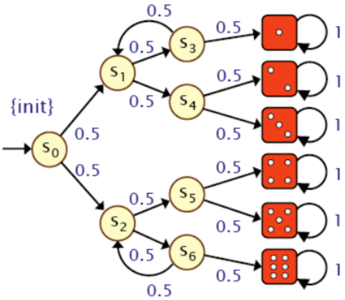


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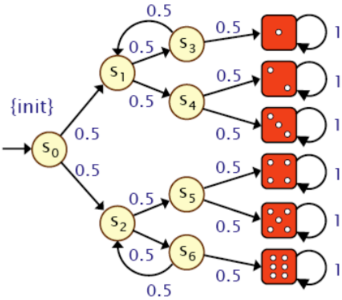


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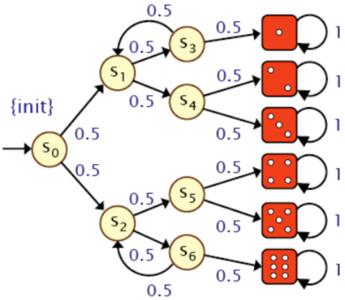


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$$x_s \;=\; \sum_{t \in S \setminus G} \mathbf{P}(s, t) \cdot x_t \;+\; \sum_{u \in G} \mathbf{P}(s, u)$$

In the previous characterisation we basically set:

- $S_{=1} \; = \; G$
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These sets can easily be determined in linear time by a graph analysis.

# Iteratively computing reachability probabilities

## Theorem

The vector $\mathbf{x} = \left( Pr(s \models \overline{F} \,\mathrm{U}\, G) \right)_{s \in S_?}$ is the *unique* solution of:

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where $\overline{F} \,\mathrm{U}^{\leqslant n} G$ contains those paths that reach $G$ via $\overline{F}$ within $n$ steps.

# Proof

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There are various algorithms to compute $\mathbf{x} = \lim_{n \to \infty} \mathbf{x}^{(n)}$ where:

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Then:

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The Power method computes vectors $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ and aborts if:

$$\max_{s\in S_?} \left| x_s^{(n+1)} - x_s^{(n)} \right| < \varepsilon \quad \text{for some small tolerance } \varepsilon$$

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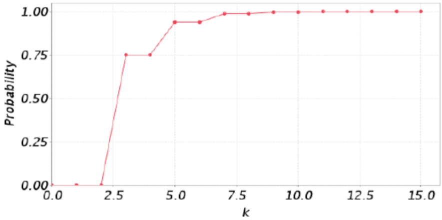
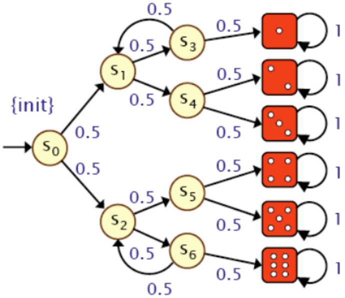
Alternatives: e.g., Jacobi or Gauss-Seidel, successive overrelaxation (SOR).

# Example: Knuth's die

- Let $G = \{1, 2, 3, 4, 5, 6\}$
- Then $Pr(s_0 \models \lozenge G) = 1$
- And $Pr(s_0 \models \lozenge^{\leqslant k} G)$ for $k \in \mathbb{N}$ is given by:



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When considering $\Theta_n^{\mathcal{D}}$ as vector $(\Theta_n^{\mathcal{D}})_{t \in S}$ we have:

$$\Theta_n^{\mathcal{D}} \;=\; \iota_{\mathrm{init}} \cdot \underbrace{\mathbf{P} \cdot \mathbf{P} \cdot \ldots \cdot \mathbf{P}}_{n \text{ times}} \;=\; \iota_{\mathrm{init}} \cdot \mathbf{P}^n.$$

Computation: $\Theta_0^{\mathcal{D}} = \iota_{\mathrm{init}}$ and $\Theta_{n+1}^{\mathcal{D}} = \Theta_n^{\mathcal{D}} \cdot \mathbf{P}$ for $n \geqslant 0$.

# Reachability probabilities vs. transient probabilities

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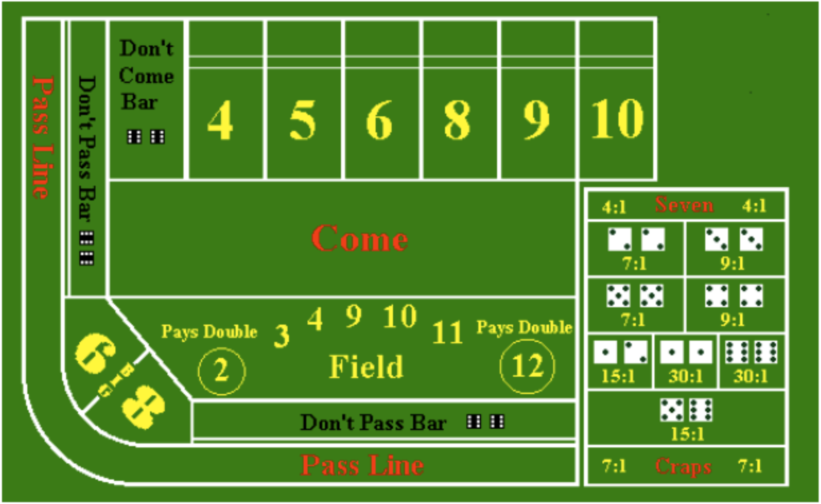
# Spare time tonight? Play Craps!



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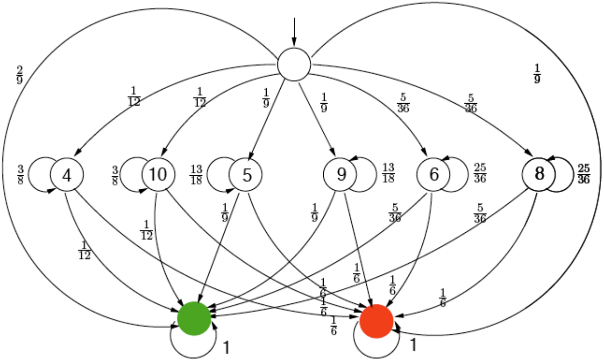


What is the probability to win the Craps game?

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2. Events $\lozenge\, G$, $\square\lozenge\, G$ and $\overline{F}\,\mathrm{U}\, G$ are measurable.
3. Reachability probabilities = unique solution of linear equation system.
4. Bounded reachabilities = transient probabilities in a modified DTMC.