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The Satisfiability Problem for Probabilistic CTL

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My Paper

This presentation is based on the paper

The Satisfiability Problem for Probabilistic CTL
By Thomáš Brázdil, Vojtěch Forejt, Jan Křetínský and Antonín Kučera from Masaryk University Brno, Czech Republic
Published in 2008 on the 23rd Annual IEEE Symposium on Logic in Computer Science.

PCTL

Probabilistic Computation Tree Logic

The syntax of PCTL state formulas is

$$\Phi := \quad \text{true} \quad | \quad a \quad | \quad \Phi_1 \wedge \Phi_2 \quad | \quad \neg\Phi \quad | \quad \mathbb{P}_J(\varphi),$$

where $a \in AP$ is an atomic proposition and φ is a CTL path formula. J is of the form $\bowtie p$ with $\bowtie \in \{<, >, \leqslant, \geqslant\}$ and $p \in [0, 1]$.
PCTL path formulas follow the syntax

$$\varphi := \quad \bigcirc \Phi \quad | \quad \Phi_1 \,\mathbf{U}\, \Phi_2,$$

where Φ, Φ_1 and Φ_2 are state formulas.

PCTL

Qualitative PCTL

The qualitative fragment of PCTL is identical to PCTL, but the probability operator $\mathbb{P}_J(\varphi)$ is restricted to $J \in \{= 0, > 0, < 1, = 1\}$.

For example:

- $\mathbb{P}_{=0}\left(\Phi_1 \,\mathsf{U}\, \Phi_2\right)$,
- $\mathbb{P}_{>0}\left(\Phi_1 \,\mathsf{U}\, \Phi_2\right)$,
- $\mathbb{P}_{<1}\left(\Phi_1 \,\mathsf{U}\, \Phi_2\right)$,
- $\mathbb{P}_{=1}\left(\Phi_1 \,\mathsf{U}\, \Phi_2\right)$.

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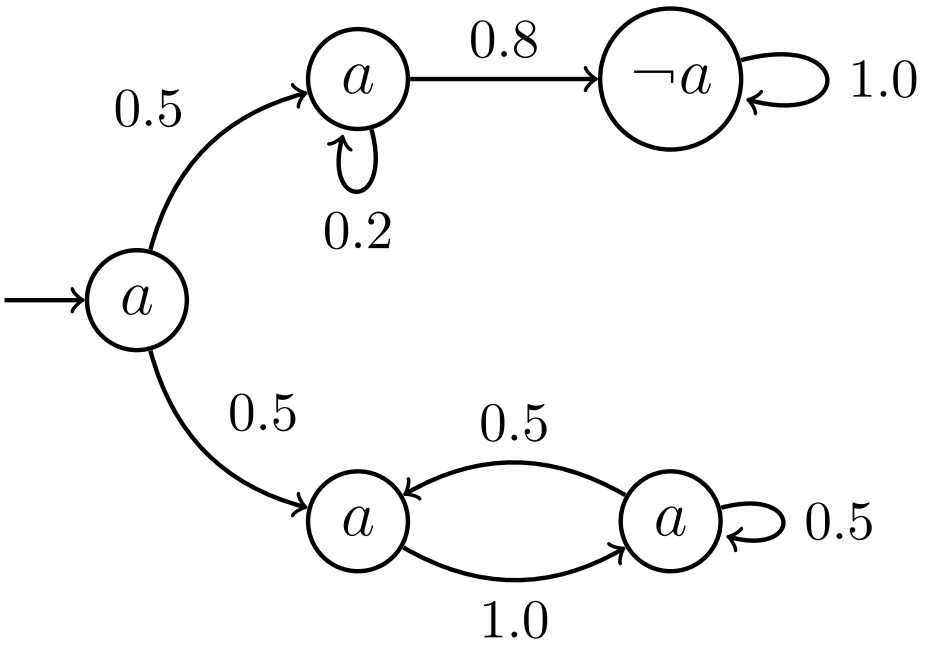
Markov Chains

$$\mathcal{M} := (S, \mathbf{P}, s_{init}, AP, L)$$



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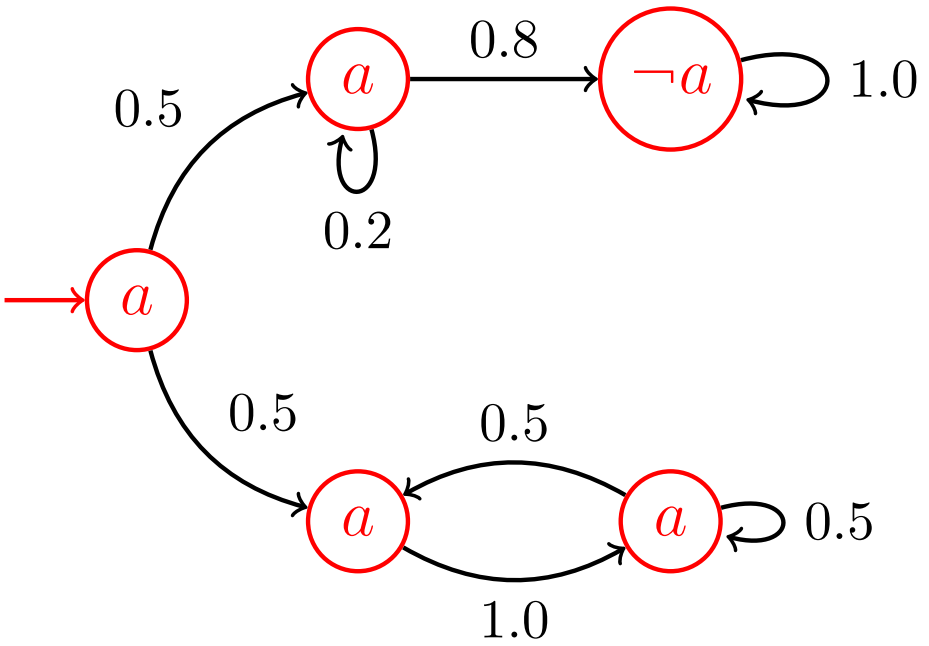

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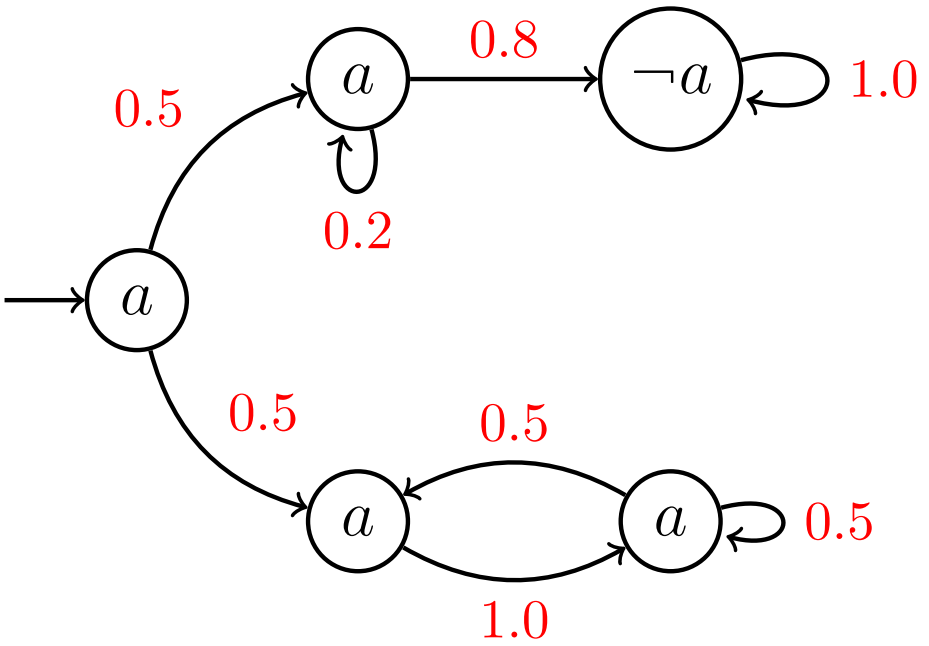

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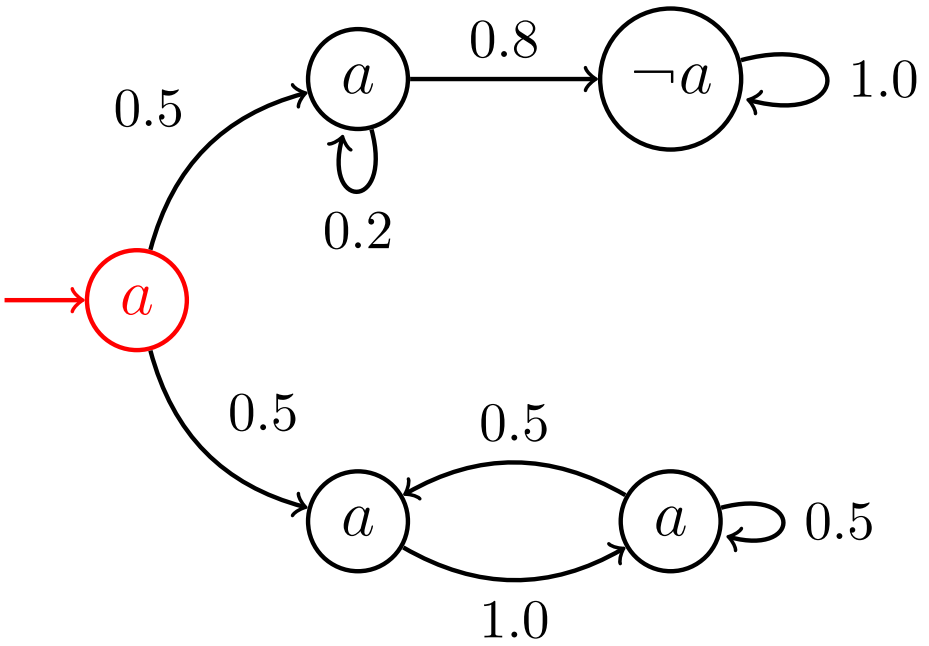

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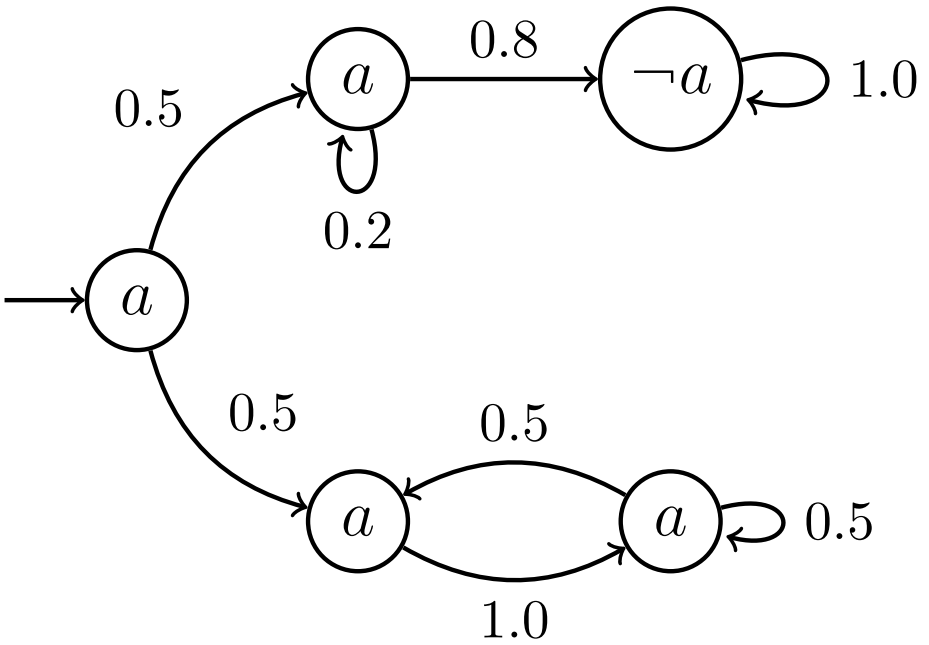

$$\Phi := \mathbb{P}_{>0}\,(a\,\mathbf{U}\,\neg a)$$

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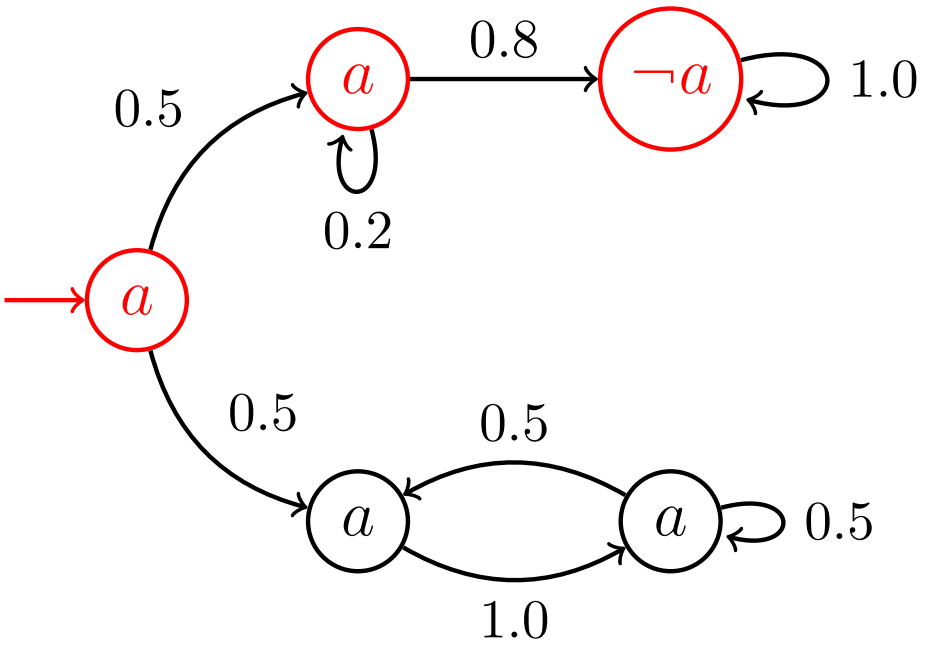
Markov Chains

$$\mathcal{M} := (S, \mathbf{P}, s_{init}, AP, L)$$



$$\Phi := \mathbb{P}_{>0}\,(a\,\mathbf{U}\,\neg a)$$

Satisfiability

The Satisfiability Problem

input	a formula Φ in a logic
question	does there exist a model $\mathcal{M}$ such that $\mathcal{M} \models \Phi$
output	*satisfiable* and a suitable model, or *unsatisfiable*

Well known example: 3-SAT for boolean logic formulas in conjunctive normal form.

$$\Phi = (\neg a \vee b \vee c) \wedge (a \vee \neg b \vee \neg c)$$

Satisfiability

The Satisfiability Problem for Probabilistic CTL

input a PCTL formula Φ

question does there exist a Markov–chain $\mathcal{M}$ such that $\mathcal{M}_s \models \Phi$ for some state $s \in \mathcal{M}$

output *satisfiable* and a suitable Markov–chain, or *unsatisfiable*

- Inverse problem to model checking.
- Very similar to the Satisfiability Problem for CTL.

Satisfiability of CTL

The Satisfiability Problem for CTL

Shown by Emerson and Halpern in 1982:

- The satisfiability problem for CTL is EXPTIME-complete.
- If a CTL formula is satisfiable, there exists a finite model with **bounded size** (small model property).

Satisfiability of PCTL

The Satisfiability Problem for PCTL

- No small model property, even infinite models:

$$\Box^{>0}\left(\neg a \wedge \Diamond^{>0} a\right)$$

Intuition:
There exists a path on which "$\Box\neg a$" holds, but a is always reachable.

Overview

Problem: For a given PCTL formula Φ — is it satisfiable? And if so, give a finite representation of the model.

1. Deduce a model–like structure from the initial formula Φ.
2. Iterate certain **logical checks** for consistency, prune parts that do not fit.
3. On convergence: check whether the initial formula is satisfied in a state.
4. If yes: return abstract model description, or
5. if no: return unsatisfiable.

The closure of a formula

Fischer-Ladner closure for PCTL

This closure of Φ contains:

- Φ itself,
- negations,
- all particles of a formula,
- all implications of a formula.

Eligible sets

- taking arbitrary subsets of the closure yields unfeasible sets
- therefore: enforce consistency and completeness

Eligible sets

A subset S of the closure of Φ is consistent or eligible iff for every $\Psi \in Cl(\Phi)$ it holds that:

- either Ψ or $\neg\Psi$ is in S,
- and logical implications are fulfilled.

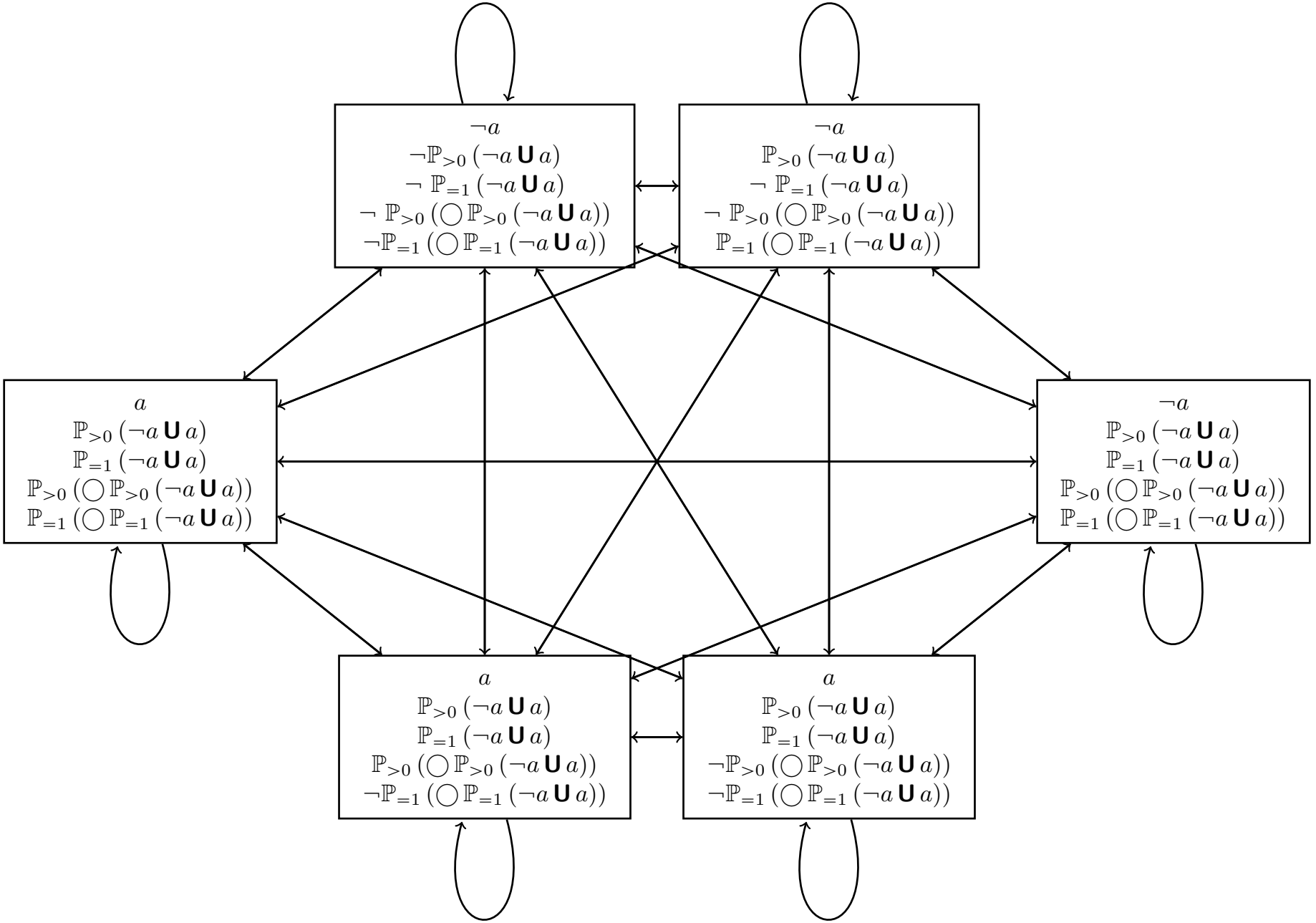
Eligible sets - Example

$$\Phi := \mathbb{P}_{=1}\left(\neg a \,\mathbf{U}\, a\right).$$

The closure $\mathrm{Cl}\left(\Phi\right)$ contains:

1. $\mathbb{P}_{=1}(\neg a \mathbf{U} a)$ (Φ itself)
2. $\neg a$
3. a
4. $\mathbb{P}_{=1}(\bigcirc \mathbb{P}_{=1}(\neg a \mathbf{U} a))$
5. $\mathbb{P}_{>0}(\neg a \mathbf{U} a)$
6. $\mathbb{P}_{>0}(\bigcirc \mathbb{P}_{>0}(\neg a \mathbf{U} a))$

(plus the negated formulas)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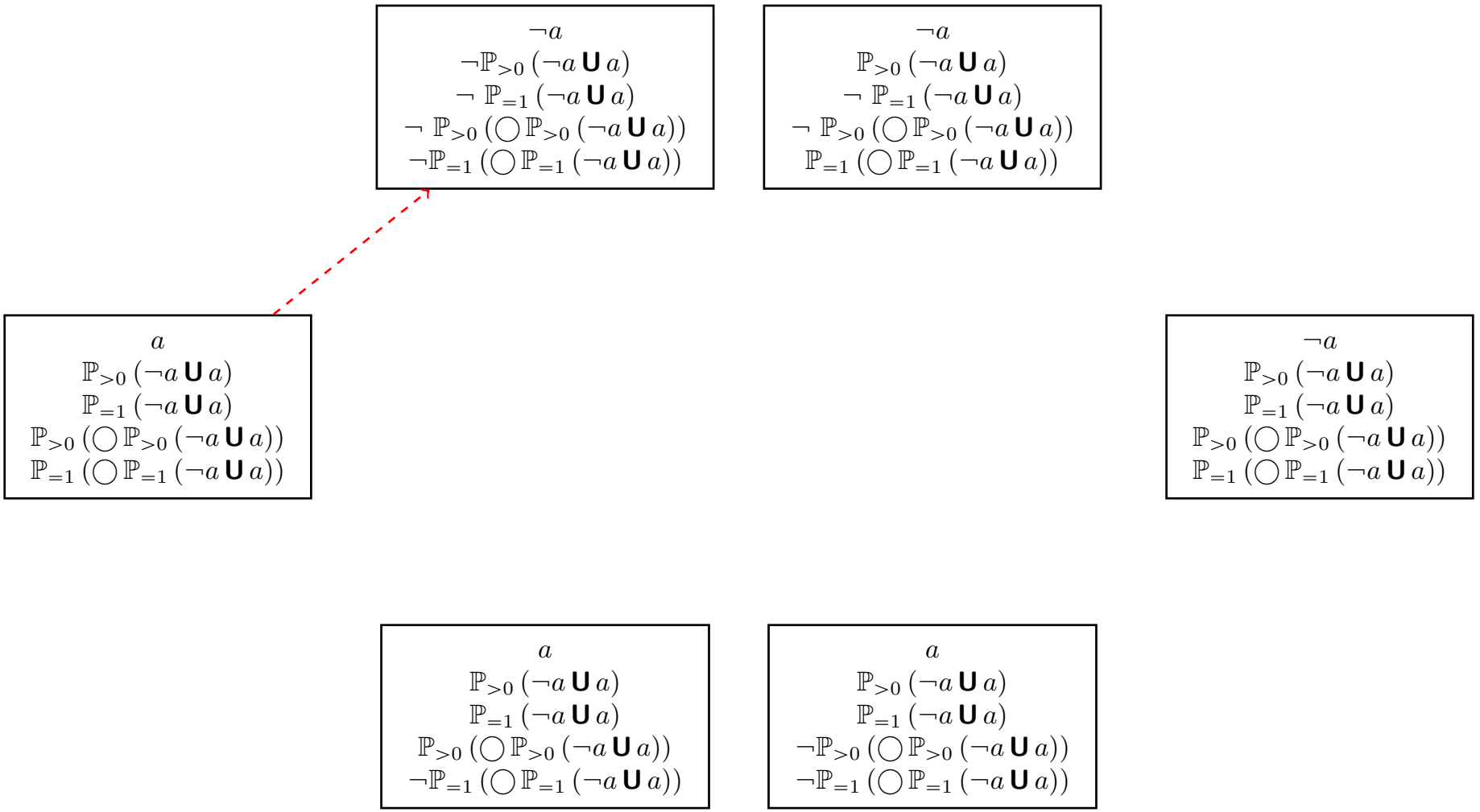
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Overview

Problem: For a given PCTL formula Φ — is it satisfiable? And if so, give a finite representation of the model.

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3. On convergence: check whether the initial formula is satisfied in a state.
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5. if no: return unsatisfiable.

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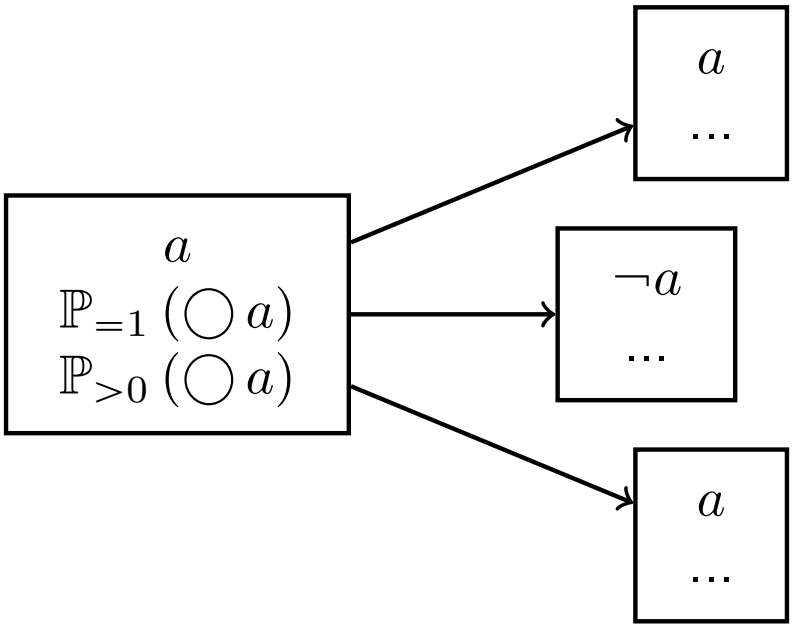
Logical checks

Do $\Phi_1 = \mathbb{P}_{>0}\left(\bigcirc a\right)$ and $\Phi_2 = \mathbb{P}_{=1}\left(\bigcirc a\right)$ hold?



Static and iterative checks

- Static check: remove transition
- Iterative check: remove state

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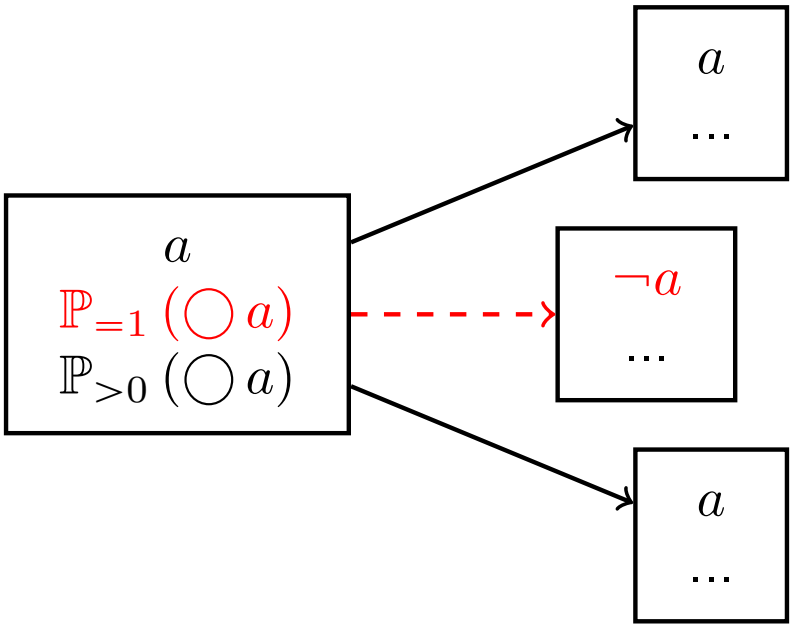

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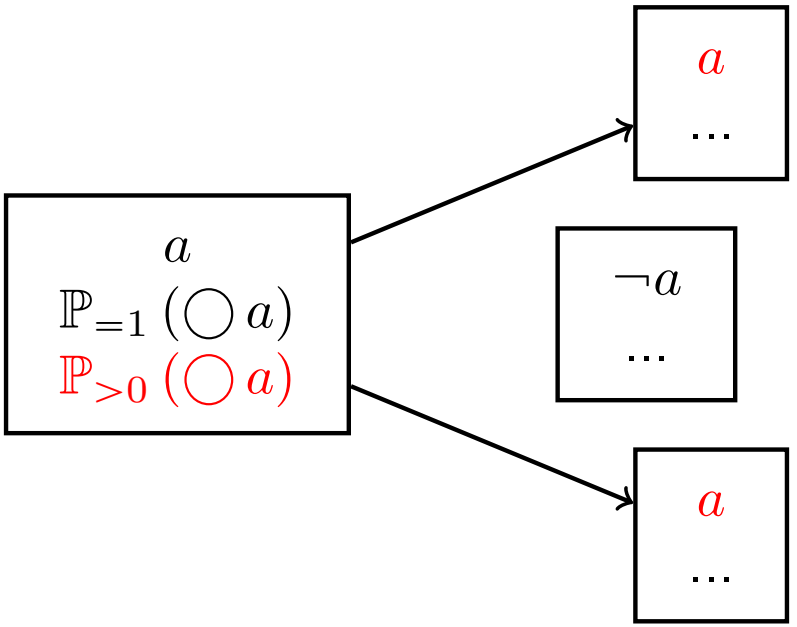

Static and iterative checks

- Static check: remove transition
- Iterative check: remove state

Intuition

- The set of eligible states contains all possible states for Φ.
- Using an uniform distribution we could define a Markov chain.
- But the amount of eligible states is at most exponential in $|\Phi|$.

How can we algorithmically find a finite representation of an infinite model?

Intuition

- The set of eligible states contains all possible states for Φ.
- Using an uniform distribution we could define a Markov chain.
- But the amount of eligible states is at most exponential in $|\Phi|$.

How can we algorithmically find a finite representation of an infinite model?

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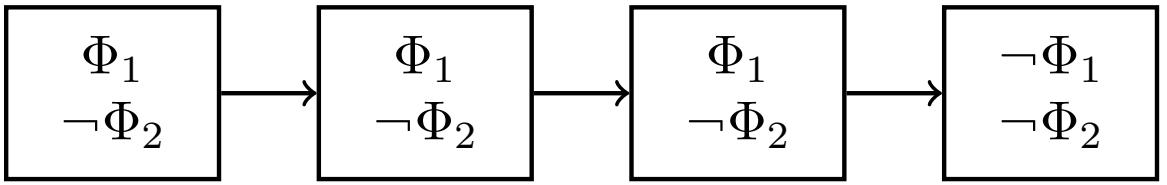
Infinite Models

Only one kind of problematic formulas:

$$\Phi := \neg \mathbb{P}_{=1} \left(\Phi_1 \, \mathsf{U} \, \Phi_2 \right)$$

- Either a finite path exists:

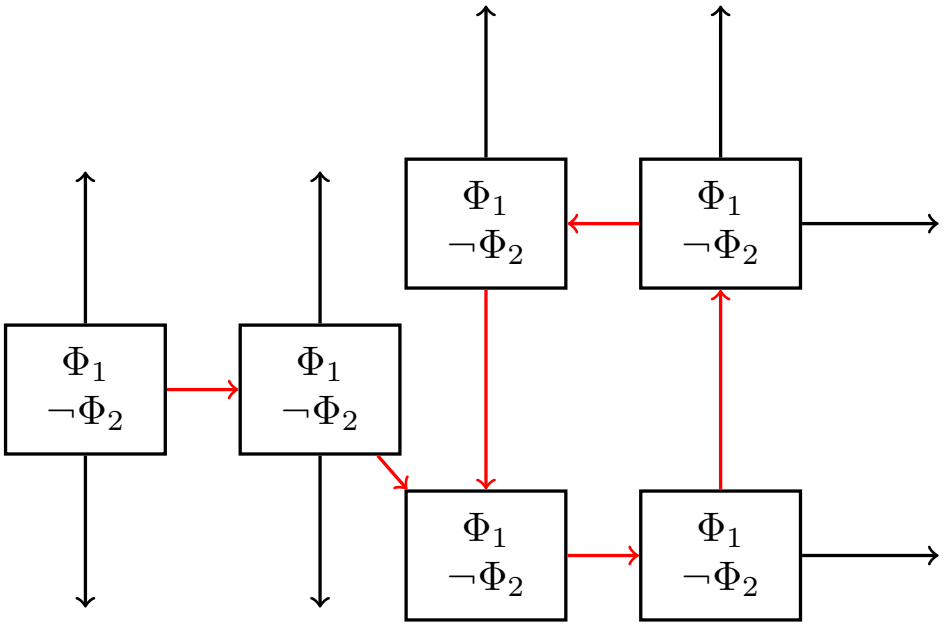


- or it gets complicated.

Infinite Models

$$\Phi := \neg \mathbb{P}_{=1} \left(\Phi_1 \, \mathbf{U} \, \Phi_2 \right)$$

- or a set of runs $\mathcal{R}$ exists for which $\mathcal{P}(\mathcal{R}) > 0$ and $\Phi_1 \wedge \neg \Phi_2$ holds in every state.



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Marked graphs

Marked graphs

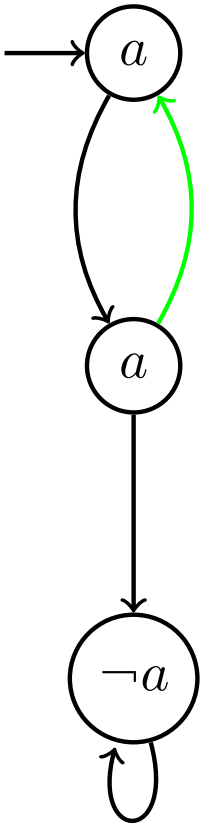
A marked graph is a triple $\mathcal{G} = (G, \hookrightarrow, L)$ where G is a finite set of nodes, $\hookrightarrow \subseteq G \times G$ is a relation, and $L \subseteq \hookrightarrow$ a subset of marked transitions.

If a state has marked and unmarked transitions, all unmarked transitions get a probability p and all marked transitions a probability p' such that

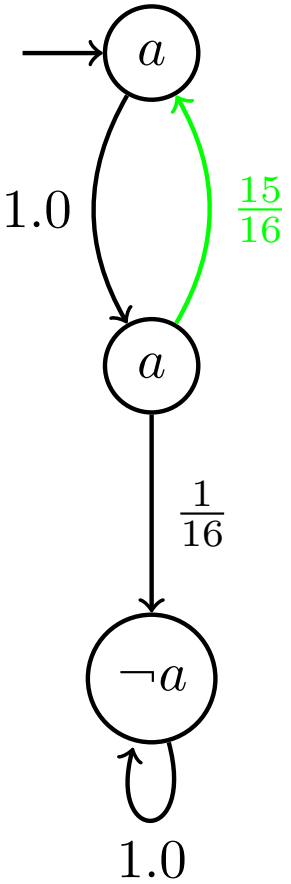
$$\sum_{w \to w'} p' = 1 - \left(\frac{1}{4}\right)^{len(w)},$$

$$\sum_{w \to w'} p = \left(\frac{1}{4}\right)^{len(w)}.$$

Marked graphs - Example



Marked graphs - Example



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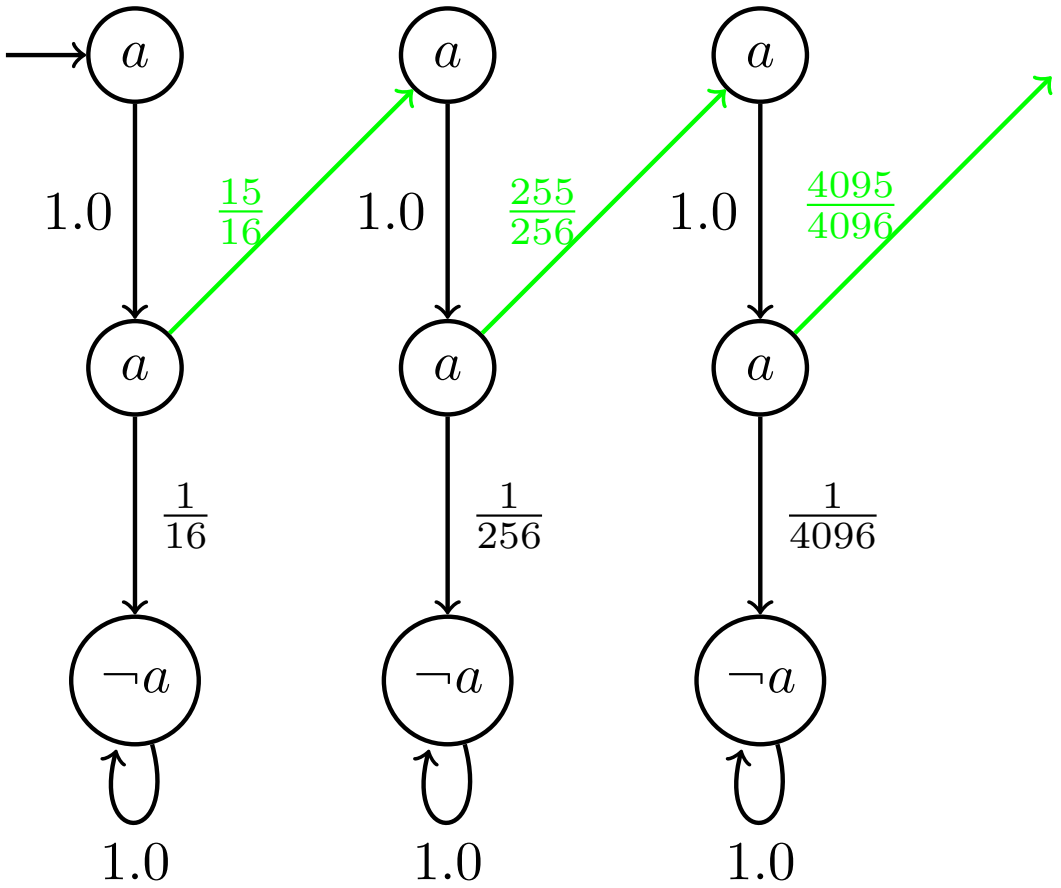
Marked graphs - Example



Pseudo–models

Pseudo–models

$\mathcal{A} := (A, \rightarrow)$, where

- A is a set of eligible subsets of the closure of Φ,
- each state $s \in A$ is labeled with the elements of the closure which it contains, and
- $\rightarrow \subseteq A \times A$ is a relation.

All formulas except for $\neg \mathbb{P}_{=1} (\Phi_1 \, \mathbf{U} \, \Phi_2)$ have to be fulfilled by the underlying model.

Witnesses

For $\neg\mathbb{P}_{=1}\left(\Phi_1 \,\mathbf{U}\, \Phi_2\right)$ to hold in s we need either

- a finite path starting in s which ends in a state with $\neg\Phi_1$ and $\neg\Phi_2$, or
- a set of infinite paths with a non–zero probability on which Φ_1 and $\neg\Phi_2$ holds.

Witnesses

A witness for a formula $\neg\mathbb{P}_{=1}\left(\Phi_1 \,\mathbf{U}\, \Phi_2\right) \in \mathrm{Cl}\left(\Phi\right)$ is a sub–pseudo–structure $\mathcal{B} \subseteq \mathcal{A}$ such that

- $\mathcal{B}$ is strongly connected and
- for every $s \in B$ we have that $\neg\Phi_2 \in s$.

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3. On convergence: check whether the initial formula is satisfied in a state.
4. If yes: return abstract model description, or
5. if no: return unsatisfiable.

The Algorithm I

1. Decision: Finite or Infinite Model?
2. Structure:
   - Two preliminary static checks for $(\neg \mathbb{P}_{>0}(\bigcirc \Phi_1), \mathbb{P}_{=1}(\bigcirc \Phi_1))$
   - Loop (remove states):
     - Check *path existence* for formulas like $\mathbb{P}_{>0}(\bigcirc \Phi_1), \neg \mathbb{P}_{=1}(\bigcirc \Phi_1), \mathbb{P}_{>0}(\Phi_1 \,\mathbf{U}\, \Phi_2)$,
     - check the existence of *problematic BSCCs* for all $\mathbb{P}_{=1}(\Phi_1 \,\mathbf{U}\, \Phi_2)$,
     - check formulas of the form $\neg \mathbb{P}_{=1}(\Phi_1 \,\mathbf{U}\, \Phi_2)$ (path or witness),
     - remove unreachable states and those with no successors.
   - Check for convergence (no states were removed in the last iteration).

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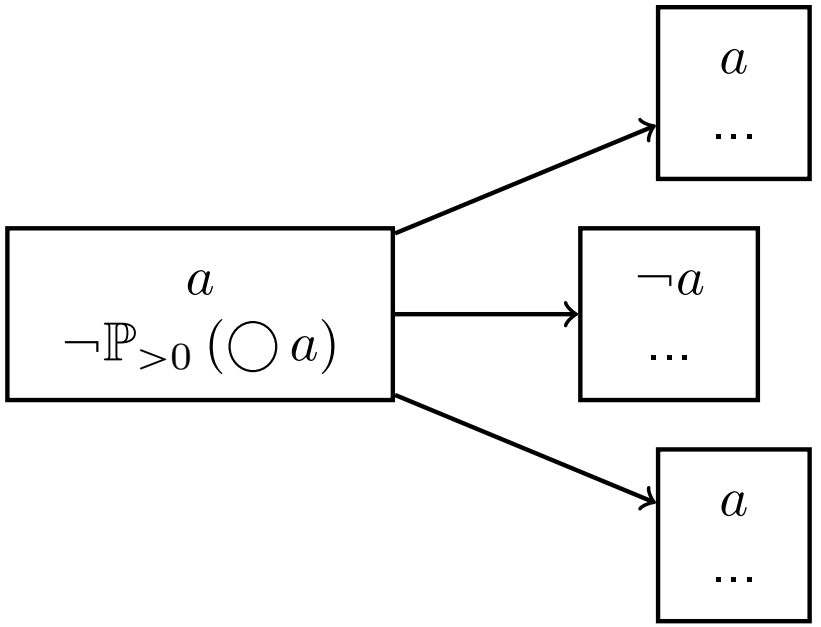
The Algorithm II

Preliminary static checks for $(\neg \mathbb{P}_{>0} (\bigcirc \Phi_1), \mathbb{P}_{=1} (\bigcirc \Phi_1))$:



For each state s, remove all transitions which invalidate the formula.

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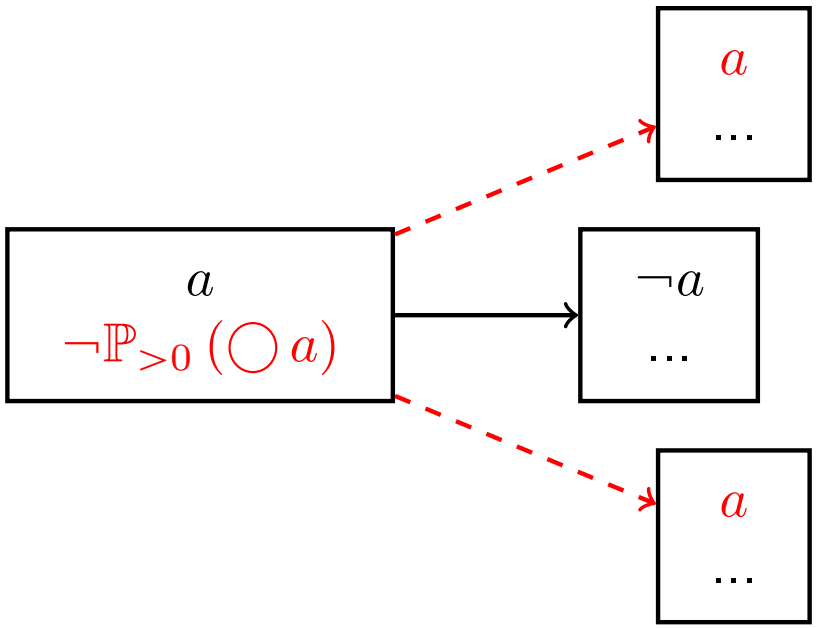
The Algorithm II

Preliminary static checks for $(\neg\mathbb{P}_{>0}(\bigcirc\Phi_1), \mathbb{P}_{=1}(\bigcirc\Phi_1))$:



For each state s, remove all transitions which invalidate the formula.

The Algorithm III

Path existence

A state s with a formula Φ

- $\mathbb{P}_{>0}\left(\bigcirc \Phi_1\right)$: "simple path" – check successors.
- $\neg\mathbb{P}_{=1}\left(\bigcirc \Phi_1\right)$: "simple path" – check successors.
- $\mathbb{P}_{>0}\left(\Phi_1 \,\mathbf{U}\, \Phi_2\right)$: check using PCTL model checking techniques.

If a check fails, i.e. no path exists: remove the state s.

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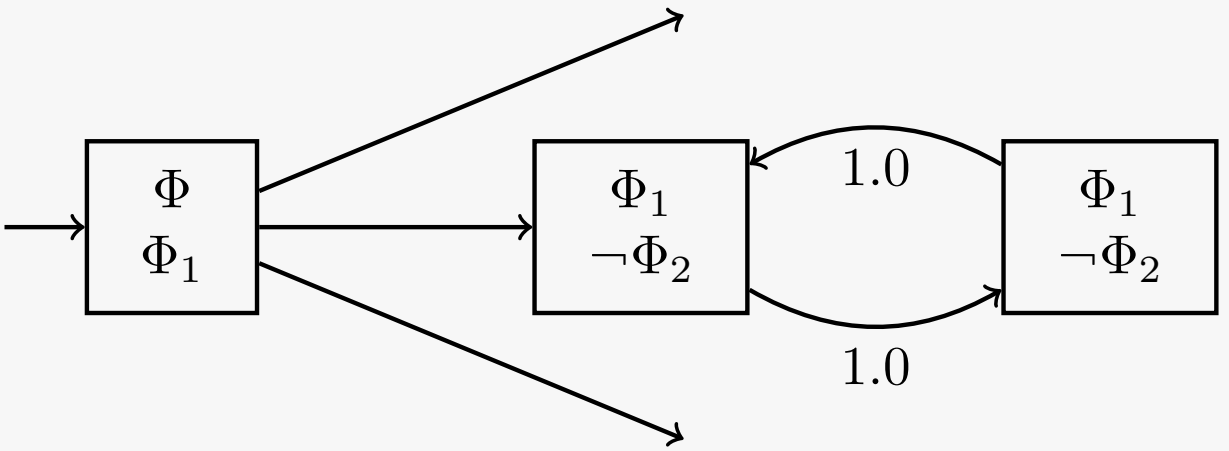
The Algorithm IV

Problematic BSCCs for $\mathbb{P}_{=1}(\Phi_1 \,\mathbf{U}\, \Phi_2)$

A state s with a formula $\Phi = \mathbb{P}_{=1}(\Phi_1 \,\mathbf{U}\, \Phi_2)$:

- Similar to $\neg\mathbb{P}_{=1}(\Phi_1 \,\mathbf{U}\, \Phi_2)$
- but we do **not** allow infinite counterexamples here
- therefore: check and remove any existing problematic BSCC.



The Algorithm V

Finite or infinite models

A state s with a formula $\Phi = \neg\mathbb{P}_{=1}(\Phi_1 \,\mathbf{U}\, \Phi_2)$

- for finite models: check for a finite path $\pi = s_0, ..., s_n$ with $s = s_0$ and $\Phi_1 \wedge \neg\Phi_2$ on all states $s_i, i < n$ and $\neg\Phi_1 \wedge \neg\Phi_2$ on state s_n.
- for infinite models: check for a finite path, if no such path exists check whether a witness for Φ is reachable.

If a check fails, i.e. no path exists: remove the state s.

The Algorithm VI

Final steps

- If no state was removed in an iteration, the model is finished.
- If all states containing Φ have been removed, there exists no (finite) model for Φ.

The resulting pseudo–model can be converted to a Markov chain.

The Algorithm VII

Complexity

The satisfiability problem and the finite-satisfiability problem for qualitative PCTL are EXPTIME-complete.

- The amount of eligible states is exponential in $|\Phi|$.
- Model-checking formulas during iterations is polynomial in the size of Φ and $\mathcal{A}$.
- At most $|\mathcal{A}|$ iterations.

Proof of hardness similar to the proof for CTL.

What we have seen

- A very simple approach for qualitative PCTL satisfiability checking
- A representation for infinite-state models
- An iterative algorithm using these ideas

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Questions?

Thank you for your Attention!

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Sources

[1] Tomás Brázdil, Vojtech Forejt, J Kretinsky, and Antonín Kucera. The satisfiability problem for probabilistic ctl. In Logic in Computer Science, 2008. LICS'08. 23rd Annual IEEE Symposium on, pages 391–402. IEEE, 2008.

[2] E. Allen Emerson and Joseph Y. Halpern. Decision procedures and expressiveness in the temporal logic of branching time. In Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing, STOC '82, pages 169–180, New York, NY, USA, 1982. ACM.

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The closure of a formula

Fischer-Ladner closure for PCTL:

1. $\Phi \in \mathrm{Cl}(\Phi)$
2. $\Phi_1 \in \mathrm{Cl}(\Phi) \Rightarrow \neg\Phi_1 \in \mathrm{Cl}(\Phi)$
3. $\Phi_1 \wedge \Phi_2 \in \mathrm{Cl}(\Phi) \Rightarrow \Phi_1 \in \mathrm{Cl}(\Phi)$ and $\Phi_2 \in \mathrm{Cl}(\Phi)$
4. $\mathbb{P}_{\bowtie}(\bigcirc \Phi_1) \in \mathrm{Cl}(\Phi) \Rightarrow \Phi_1 \in \mathrm{Cl}(\Phi)$
5. $\mathbb{P}_{\bowtie}(\Phi_1 \,\mathsf{U}\, \Phi_2) \in \mathrm{Cl}(\Phi) \Rightarrow \Phi_1 \in \mathrm{Cl}(\Phi)$ and $\Phi_2 \in \mathrm{Cl}(\Phi)$ and $\mathbb{P}_{\bowtie}(\bigcirc \mathbb{P}_{\bowtie}(\Phi_1 \,\mathsf{U}\, \Phi_2)) \in \mathrm{Cl}(\Phi)$
6. $\mathbb{P}_{=1}(\Phi_1 \,\mathsf{U}\, \Phi_2) \in \mathrm{Cl}(\Phi) \Rightarrow \mathbb{P}_{>0}(\Phi_1 \,\mathsf{U}\, \Phi_2) \in \mathrm{Cl}(\Phi)$

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Eligible sets

- taking arbitrary subsets of the closure yields unfeasible sets
- enforce consistency and completeness

A subset S of the closure of Φ is consistent or eligible iff for every $\Psi \in Cl(\Phi)$ it holds that:

- Either Ψ or $\neg\Psi$ is in S.
- For $\Psi_1 \wedge \Psi_2 \in S$, we have that both Ψ_1 and Ψ_2 are in S.
- For $\neg(\Psi_1 \wedge \Psi_2) \in S$, we have that either $\neg\Psi_1$ or $\neg\Psi_2$ is in S.
- For $\mathbb{P}_{\bowtie}(\Psi_1 \,\mathbf{U}\, \Psi_2) \in S$, we have that either Ψ_2 or $\mathbb{P}_{\bowtie}(\bigcirc \mathbb{P}_{\bowtie}(\Psi_1 \,\mathbf{U}\, \Psi_2))$ is in S.
- For $\neg\mathbb{P}_{\bowtie}(\Psi_1 \,\mathbf{U}\, \Psi_2) \in S$, we have that either $\neg\Psi_1, \neg\Psi_2$ or $\neg\Psi_2, \neg\mathbb{P}_{\bowtie}(\bigcirc \mathbb{P}_{\bowtie}(\Psi_1 \,\mathbf{U}\, \Psi_2))$ are in S.