Static Program Analysis

Lecture 5: Dataflow Analysis IV (Worklist Algorithm & MOP Solution)

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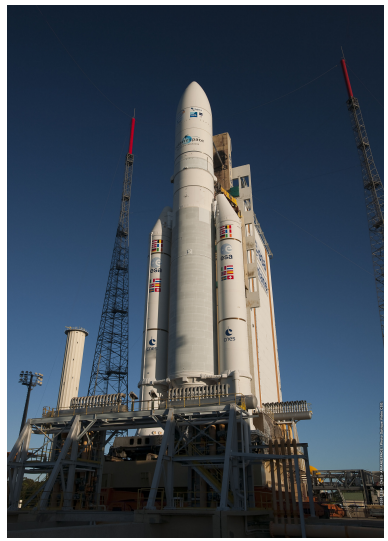
Wanted: Software Engineering HiWis

- What we offer: work in
  - EU project D-MILS
    - Dependability and Security of Distributed Information and Communication Infrastructures
    - http://www.d-mils.org/
    - Goal: [design and] implementation of high-level specification language
  - ESA project CATSY
    - Catalogue of System and Software Properties
    - Successor of COMPASS project (http://compass.informatik.rwth-aachen.de)
    - goal: support early V & V activities in model-based system development
- What we expect: prospective candidates
  - like formal methods (model checking, program/model transformations)
  - program efficiently (Python)
  - work 9–19 hrs/week
- Contact: Thomas Noll (noll@cs.rwth-aachen.de)



Outline

The Fixpoint Theorem

Alfred Tarski (1901–1983)

Bronislaw Knaster (1893–1990)

Theorem (Fixpoint Theorem by Tarski and Knaster)

Let $(D, \sqsubseteq)$ *be a complete lattice satisfying ACC and* $\Phi : D \to D$ *monotonic. Then*

$$\text{fix}(\Phi) := \bigsqcup \left\{ \Phi^k(\bot) \mid k \in \mathbb{N} \right\}$$

is the least fixpoint of Φ *where*

$$\Phi^0(d) := d \text{ and } \Phi^{k+1}(d) := \Phi(\Phi^k(d)).$$

Function requirements for dataflow analysis

All transfer functions must be a monotonic

Dataflow Systems

Definition (Dataflow system)

A dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ consists of

- a finite set of (program) labels Lab (here: Lab_c),
- a set of extremal labels $E \subseteq Lab$ (here: $\{\text{init}(c)\}$ or $\text{final}(c)$),
- a flow relation $F \subseteq Lab \times Lab$ (here: $\text{flow}(c)$ or $\text{flow}^R(c)$),
- a complete lattice $(D, \sqsubseteq)$ satisfying ACC
  (with LUB operator $\bigsqcup$ and least element $\bot$),
- an extremal value $\iota \in D$ (for the extremal labels), and
- a collection of monotonic transfer functions $\{\varphi_l \mid l \in Lab\}$ of type $\varphi_l : D \to D$.

Dataflow Systems and Fixpoints

Definition (Dataflow equation system)

Given: dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$, $Lab = \{1, ..., n\}$ (w.l.o.g.)

- S determines the equation system (where $l \in Lab$)

$$\mathsf{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

- $(d_1, \ldots, d_n) \in D^n$ is called a solution if

$$d_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(d_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

- S determines the transformation

$$\Phi_S : D^n \to D^n : (d_1, \ldots, d_n) \mapsto (d'_1, \ldots, d'_n)$$

where

$$d'_l := \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(d_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

Corollary

$(d_1, \ldots, d_n) \in D^n$ solves the equation system iff it is a fixpoint of Φ_S

Solving Dataflow Problems by Fixpoint Iteration

Remarks:

- $(D, \sqsubseteq)$ being a complete lattice ensures that Φ_S is well defined
- Since $(D, \sqsubseteq)$ is a complete lattice satisfying ACC, so is $(D^n, \sqsubseteq^n)$ (where $(d_1, \ldots, d_n) \sqsubseteq^n (d'_1, \ldots, d'_n)$ iff $d_i \sqsubseteq d'_i$ for every $1 \leq i \leq n$)
- Monotonicity of transfer functions φ_l in $(D, \sqsubseteq)$ implies monotonicity of Φ_S in $(D^n, \sqsubseteq^n)$ (since $\bigsqcup$ also monotonic)
- Thus the (least) fixpoint is effectively computable by iteration:

$$\mathrm{fix}(\Phi_S) = \bigsqcup \{\Phi_S^k(\bot_{D^n}) \mid k \in \mathbb{N}\}$$

  where $\bot_{D^n} = (\underbrace{\bot_D, \ldots, \bot_D}_{n \text{ times}})$

- If height of $(D, \sqsubseteq)$ is m
  $\Longrightarrow$ height of $(D^n, \sqsubseteq^n)$ is $m \cdot n$
  $\Longrightarrow$ fixpoint iteration requires at most $m \cdot n$ steps

Outline

Uniqueness of Solutions I

Observation: (non-minimal) solutions of dataflow equation systems are not always unique.

Example 5.1 (Available Expressions)

```
[z := x+y]^1;
while [true]^2 do
  [skip]^3;
```

$$\begin{aligned} &\Longrightarrow \quad AE_1 = \emptyset \\ &\qquad\quad AE_2 = (AE_1 \cup \{\texttt{x+y}\}) \cap AE_3 \\ &\qquad\quad AE_3 = AE_2 \end{aligned}$$

$$\begin{aligned} &\Longrightarrow \quad AE_1 = \emptyset \\ &\qquad\quad AE_2 = \{\texttt{x+y}\} \cap AE_3 \\ &\qquad\quad AE_3 = AE_2 \end{aligned}$$

$\Longrightarrow$ Solutions: $AE_1 = AE_2 = AE_3 = \emptyset$ or
$AE_1 = \emptyset, AE_2 = AE_3 = \{\texttt{x+y}\}$

Here: greatest solution $\{\texttt{x+y}\}$ (maximal potential for optimisation)

Uniqueness of Solutions II

Example 5.2 (Live Variables)

$$
\begin{array}{l}
\texttt{while}\ [\texttt{x>1}]^1\ \texttt{do} \\
\quad [\texttt{skip}]^2; \\
[\texttt{x := x+1}]^3; \\
[\texttt{y := 0}]^4
\end{array}
\qquad
\begin{array}{ll}
\Longrightarrow & \mathsf{LV}_1 = \mathsf{LV}_2 \cup (\mathsf{LV}_3 \cup \{\texttt{x}\}) \\
& \mathsf{LV}_2 = \mathsf{LV}_1 \cup \{\texttt{x}\} \\
& \mathsf{LV}_3 = \mathsf{LV}_4 \setminus \{\texttt{y}\} \\
& \mathsf{LV}_4 = \{\texttt{x}, \texttt{y}\} \\
\Longrightarrow & \mathsf{LV}_3 = \{\texttt{x}\} \\
\Longrightarrow & \mathsf{LV}_1 = \mathsf{LV}_2 \cup \{\texttt{x}\} \\
& \quad = \mathsf{LV}_1 \cup \{\texttt{x}\}
\end{array}
$$

$\Longrightarrow$ Solutions: $\mathsf{LV}_1 = \mathsf{LV}_2 = (\{\texttt{x}\}$ or $\{\texttt{x}, \texttt{y}\})$,
$\mathsf{LV}_3 = \{\texttt{x}\}, \mathsf{LV}_4 = \{\texttt{x}, \texttt{y}\}$

Here: least solution $\{\texttt{x}\}$ (maximal potential for optimisation)

Outline

A Worklist Algorithm I

Observation: fixpoint iteration re-computes every AI_l in every step

$\Longrightarrow$ redundant if $\mathsf{AI}_{l'}$ at no F-predecessor l' changed

$\Longrightarrow$ optimisation by worklist

Algorithm 5.3 (Worklist algorithm)

Input: *dataflow system* $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$

Variables: $W \in (Lab \times Lab)^*$, $\{\mathsf{AI}_l \in D \mid l \in Lab\}$

Procedure:

$W := \varepsilon$; **for** $(l, l') \in F$ **do** $W := W \cdot (l, l')$; *% Initialise W*
for $l \in Lab$ **do** *% Initialise* AI
  if $l \in E$ **then** $\mathsf{AI}_l := \iota$ **else** $\mathsf{AI}_l := \bot_D$;
while $W \neq \varepsilon$ **do**
  $(l, l') := \mathbf{head}(W)$; $W := \mathbf{tail}(W)$;
  if $\varphi_l(\mathsf{AI}_l) \not\sqsubseteq \mathsf{AI}_{l'}$ **then** *% Fixpoint not yet reached*
    $\mathsf{AI}_{l'} := \mathsf{AI}_{l'} \sqcup \varphi_l(\mathsf{AI}_l)$;
    for $(l', l'') \in F$ **do**
      if (l', l'') *not in* W **then** $W := (l', l'') \cdot W$;

Output: $\{\mathsf{AI}_l \mid l \in Lab\}$

A Worklist Algorithm II

Example 5.4 (Worklist algorithm)

Available Expression analysis for $c =$

```
[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
```

(cf. Examples 2.9 and 4.11)

Transfer functions:

$$\varphi_1(A) = A \cup \{\texttt{a+b}\}$$
$$\varphi_2(A) = A \cup \{\texttt{a*b}\}$$
$$\varphi_3(A) = A \cup \{\texttt{a+b}\}$$
$$\varphi_4(A) = A \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}$$
$$\varphi_5(A) = A \cup \{\texttt{a+b}\}$$

Computation protocol: on the board

An "Optimisation"

Conjecture: it suffices to initialise worklist with edges leaving extremal labels (such that analysis information will propagate through CFG)

But ...

Example 5.5 (Counterexample)

Live Variables analysis for $c = [\texttt{x := 0}]^1;\ [\texttt{x := x + 1}]^2;\ [\texttt{x := 2}]^3$

Solution: $\mathsf{LV}_1 = \{\texttt{x}\}, \mathsf{LV}_2 = \emptyset, \mathsf{LV}_3 = \{\texttt{x}\}$

"Optimised" worklist algorithm:

W	LV_1	LV_2	LV_3
$(3,2)$	$\emptyset$	$\emptyset$	$\{\texttt{x}\}$
ε	$\emptyset$	$\emptyset$	$\{\texttt{x}\}$

$\Longrightarrow$ wrong result!

Correctness of Worklist Algorithm

Properties of the algorithm:

Theorem 5.6 (Correctness of worklist algorithm)

Given a dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$*, Algorithm 5.3 always terminates and computes* $\mathrm{fix}(\Phi_S)$.

Proof.

see [Nielson/Nielson/Hankin 2005, p. 75 ff] □

Outline

The MOP Solution I

- Other solution method for dataflow systems
- MOP = Meet Over all Paths
- Analysis information for block B^l
  = least upper bound over all paths leading to l
  = most precise information for l ("reference solution")

Definition 5.7 (Paths)

Let $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ be a dataflow system. For every $l \in Lab$, the set of paths up to l is given by

$$Path(l) := \{[l_1, \ldots, l_{k-1}] \mid k \geq 1, l_1 \in E, (l_i, l_{i+1}) \in F \text{ for every } 1 \leq i < k, l_k = l\}.$$

For a path $\pi = [l_1, \ldots, l_{k-1}] \in Path(l)$, we define the transfer function $\varphi_\pi : D \to D$ by

$$\varphi_\pi := \varphi_{l_{k-1}} \circ \ldots \circ \varphi_{l_1} \circ \mathrm{id}_D$$

(so that $\varphi_{[]} = \mathrm{id}_D$).

The MOP Solution II

Definition 5.8 (MOP solution)

Let $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ be a dataflow system where $Lab = \{l_1, \ldots, l_n\}$. The MOP solution for S is determined by

$$\text{mop}(S) := (\text{mop}(l_1), \ldots, \text{mop}(l_n)) \in D^n$$

where, for every $l \in Lab$,

$$\text{mop}(l) := \bigsqcup \{\varphi_\pi(\iota) \mid \pi \in Path(l)\}.$$

Remark:

- $Path(l)$ is generally infinite

$\Longrightarrow$ not clear how to compute $\text{mop}(l)$

- In fact: MOP solution generally undecidable (later)

The MOP Solution III

Example 5.9 (Live Variables; cf. Examples 2.12 and 4.12)

$$
\begin{array}{l}
c = \texttt{[x := 2]}^1\texttt{;} \\
\quad\ \texttt{[y := 4]}^2\texttt{;} \\
\quad\ \texttt{[x := 1]}^3\texttt{;} \\
\quad\ \texttt{if [y > 0]}^4\ \texttt{then} \\
\quad\quad \texttt{[z := x]}^5 \\
\quad\ \texttt{else} \\
\quad\quad \texttt{[z := y*y]}^6\texttt{;} \\
\quad\ \texttt{[x := z]}^7
\end{array}
$$

$$\Longrightarrow \mathit{Path}(1) = \{[7,5,4,3,2], [7,6,4,3,2]\}$$

$$
\begin{array}{lll}
\Longrightarrow & \mathit{mop}(1) & = \varphi_{[7,5,4,3,2]}(\iota) \sqcup \varphi_{[7,6,4,3,2]}(\iota) \\
& & = \varphi_2(\varphi_3(\varphi_4(\varphi_5(\varphi_7(\{\mathtt{x},\mathtt{y},\mathtt{z}\}))))) \sqcup \\
& & \quad\ \varphi_2(\varphi_3(\varphi_4(\varphi_6(\varphi_7(\{\mathtt{x},\mathtt{y},\mathtt{z}\})))))\\
& & = \varphi_2(\varphi_3(\varphi_4(\varphi_5(\{\mathtt{y},\mathtt{z}\})))) \sqcup \\
& & \quad\ \varphi_2(\varphi_3(\varphi_4(\varphi_6(\{\mathtt{y},\mathtt{z}\}))))\\
& & = \varphi_2(\varphi_3(\varphi_4(\{\mathtt{x},\mathtt{y}\}))) \sqcup \\
& & \quad\ \varphi_2(\varphi_3(\varphi_4(\{\mathtt{y}\}))) \\
& & = \varphi_2(\varphi_3(\{\mathtt{x},\mathtt{y}\})) \sqcup \varphi_2(\varphi_3(\{\mathtt{y}\})) \\
& & = \varphi_2(\{\mathtt{y}\}) \sqcup \varphi_2(\{\mathtt{y}\}) \\
& & = \emptyset \sqcup \emptyset \\
& & = \emptyset \qquad \text{(same as } \mathrm{fix}(\Phi_S)(1)\text{)}
\end{array}
$$

Outline

Goal of Constant Propagation Analysis

Constant Propagation Analysis

The goal of Constant Propagation Analysis is to determine, for each program point, whether a variable has a constant value whenever execution reaches that point.

Used for Constant Folding: replace reference to variable by constant value and evaluate constant expressions

Example 5.10 (Constant Propagation Analysis)

```
[x := 1]1;
[y := 1]2;
[z := 1]3;
while [z > 0]4 do
  [w := x+y]5;
  if [w = 2]6 then
    [x := y+2]7
```

- $y = z = 1$ at labels 4–7
- w, x not constant at labels 4–7
- possible optimisations:
  [true]4 [w := x+1]5 [x := 3]7

Formalising Constant Propagation Analysis I

The dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ is given by

- set of labels $Lab := Lab_c$,
- extremal labels $E := \{\text{init}(c)\}$ (forward problem),
- flow relation $F := \text{flow}(c)$ (forward problem),
- complete lattice $(D, \sqsubseteq)$ where
  - $D := \{\delta \mid \delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}\}$
    - $\delta(x) = z \in \mathbb{Z}$: x has constant value z
    - $\delta(x) = \bot$: x undefined
    - $\delta(x) = \top$: x overdefined (i.e., several possible values)
  - $\sqsubseteq \subseteq D \times D$ defined by pointwise extension of $\bot \sqsubseteq z \sqsubseteq \top$ (for every $z \in \mathbb{Z}$)

Example 5.11

$Var_c = \{\mathtt{w}, \mathtt{x}, \mathtt{y}, \mathtt{z}\}$,

$$\delta_1 = (\underbrace{\bot}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{2}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}}), \quad \delta_2 = (\underbrace{3}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{4}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}})$$

$$\Longrightarrow \quad \delta_1 \sqcup \delta_2 = (\underbrace{3}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{\top}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}})$$

Formalising Constant Propagation Analysis II

Dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ (continued):

- extremal value $\iota := \delta_\top \in D$ where $\delta_\top(x) := \top$ for every $x \in Var_c$ (i.e., every x has (unknown) default value)
- transfer functions $\{\varphi_l \mid l \in Lab\}$ defined by

$$\varphi_l(\delta) := \begin{cases} \delta & \text{if } B^l = \texttt{skip} \text{ or } B^l \in BExp \\ \delta[x \mapsto val_\delta(a)] & \text{if } B^l = (x \texttt{ := } a) \end{cases}$$

where

$$val_\delta(x) := \delta(x) \qquad val_\delta(z) := z$$

$$val_\delta(a_1\ op\ a_2) := \begin{cases} z_1\ op\ z_2 & \text{if } z_1, z_2 \in \mathbb{Z} \\ \bot & \text{if } z_1 = \bot \text{ or } z_2 = \bot \\ \top & \text{otherwise} \end{cases}$$

for $z_1 := val_\delta(a_1)$ and $z_2 := val_\delta(a_2)$

Formalising Constant Propagation Analysis III

Example 5.12

If $\delta = (\underbrace{\bot}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{2}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}})$, then

$$\varphi_l(\delta) = \begin{cases} (\underbrace{0}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{2}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}}) & \text{if } B^l = (\mathtt{w := 0}) \\ (\underbrace{3}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{2}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}}) & \text{if } B^l = (\mathtt{w := y+1}) \\ (\underbrace{\bot}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{2}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}}) & \text{if } B^l = (\mathtt{w := w+x}) \\ (\underbrace{\top}_{\mathtt{w}}, \underbrace{1}_{\mathtt{x}}, \underbrace{2}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}}) & \text{if } B^l = (\mathtt{w := z+2}) \end{cases}$$