Static Program Analysis

Lecture 4: Dataflow Analysis III (The Framework)

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(Software Modeling and Verification)

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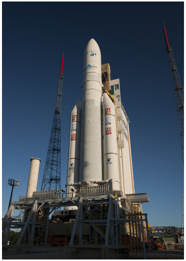
http://moves.rwth-aachen.de/teaching/ws-1415/spa/

Winter Semester 2014/15

Wanted: Software Engineering HiWis

- What we offer: work in
  - EU project D-MILS
    - Dependability and Security of Distributed Information and Communication Infrastructures
    - http://www.d-mils.org/
    - Goal: [design and] implementation of high-level specification language
  - ESA project CATSY
    - Catalogue of System and Software Properties
    - Successor of COMPASS project (http://compass.informatik.rwth-aachen.de)
    - goal: support early V & V activities in model-based system development



- What we expect: prospective candidates
  - like formal methods (model checking, program/model transformations)
  - program efficiently (Python)
  - work 9–19 hrs/week
- Contact: Thomas Noll (noll@cs.rwth-aachen.de)

Outline

Similarities Between Analysis Problems

- **Observation:** the analyses presented so far have some similarities

$\Longrightarrow$ Look for underlying framework

- **Advantage:** possibility for designing (efficient) generic algorithms for solving dataflow equations
- **Overall pattern:** for $c \in Cmd$ and $l \in Lab_c$, the analysis information (AI) is described by equations of the form

$$\mathrm{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathrm{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

where

  - the set of extremal labels, E, is $\{\mathrm{init}(c)\}$ or $\mathrm{final}(c)$
  - ι specifies the extremal analysis information
  - the combination operator, $\bigsqcup$, is $\bigcap$ or $\bigcup$
  - $\varphi_{l'}$ denotes the transfer function of block $B^{l'}$
  - the flow relation F is $\mathrm{flow}(c)$ or $\mathrm{flow}^R(c)$ $(:= \{(l', l) \mid (l, l') \in \mathrm{flow}(c)\})$

Roadmap

Goal: solve dataflow equation system by fixpoint iteration

1. Characterize solution of equation system as fixpoint of a transformation
2. Introduce partial order for comparing analysis results
3. Establish least upper bound as combination operator
4. Ensure monotonicity of transfer functions
5. Guarantee termination of fixpoint iteration by ascending chain condition
6. Optimize fixpoint iteration by worklist algorithm

Outline

Partial Orders

The domain of analysis information usually forms a partial order where the ordering relation compares the "precision" of information.

Definition (Partial order)

A partial order (PO) $(D, \sqsubseteq)$ consists of a set D, called domain, and of a relation $\sqsubseteq \subseteq D \times D$ such that, for every $d_1, d_2, d_3 \in D$,

- reflexivity: $d_1 \sqsubseteq d_1$
- transitivity: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_3 \implies d_1 \sqsubseteq d_3$
- antisymmetry: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1 \implies d_1 = d_2$

It is called total if, in addition, always $d_1 \sqsubseteq d_2$ or $d_2 \sqsubseteq d_1$.

Example

1. $(\mathbb{N}, \leq)$ is a total partial order
2. $(\mathbb{N}, <)$ is not a partial order (since not reflexive)
3. (Live Variables) $(2^{Var_c}, \subseteq)$ is a (non-total) partial order
4. (Available Expressions) $(2^{CExp_c}, \supseteq)$ is a (non-total) partial order

Upper Bounds

In the dataflow equation system, analysis information from several predecessors is combined by taking the least upper bound.

Definition ((Least) upper bound)

Let $(D, \sqsubseteq)$ be a partial order and $S \subseteq D$.

1. An element $d \in D$ is called an upper bound of S if $s \sqsubseteq d$ for every $s \in S$ (notation: $S \sqsubseteq d$).
2. An upper bound d of S is called least upper bound (LUB) or supremum of S if $d \sqsubseteq d'$ for every upper bound d' of S (notation: $d = \bigsqcup S$).

Example

1. $S \subseteq \mathbb{N}$ has a LUB in $(\mathbb{N}, \leq)$ iff it is finite
2. (Live Variables) $(D, \sqsubseteq) = (2^{Var_c}, \subseteq)$. Given $V_1, \ldots, V_n \subseteq Var_c$,
$$\bigsqcup\{V_1, \ldots, V_n\} = \bigcup\{V_1, \ldots, V_n\}$$
3. (Avail. Expr.) $(D, \sqsubseteq) = (2^{CExp_c}, \supseteq)$. Given $A_1, \ldots, A_n \subseteq CExp_c$,
$$\bigsqcup\{A_1, \ldots, A_n\} = \bigcap\{A_1, \ldots, A_n\}$$

Complete Lattices

Since $\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\}$ can contain arbitrary elements, the existence of least upper bounds must be ensured for arbitrary subsets.

Definition (Complete lattice)

A complete lattice is a partial order $(D, \sqsubseteq)$ such that all subsets of D have least upper bounds. In this case,

$$\perp := \bigsqcup \emptyset$$

denotes the least element of D.

Example

1. $(\mathbb{N}, \leq)$ is not a complete lattice as, e.g., $\mathbb{N}$ does not have a LUB
2. (Live Variables)
   $(D, \sqsubseteq) = (2^{Var_c}, \subseteq)$ is a complete lattice with $\perp = \emptyset$
3. (Available Expressions)
   $(D, \sqsubseteq) = (2^{CExp_c}, \supseteq)$ is a complete lattice with $\perp = CExp_c$

Chains

Chains are generated by the approximation of the analysis information in the fixpoint iteration.

Definition (Chain)

Let $(D, \sqsubseteq)$ be a partial order.

- A subset $S \subseteq D$ is called a chain in D if, for every $d_1, d_2 \in S$,
$$d_1 \sqsubseteq d_2 \text{ or } d_2 \sqsubseteq d_1$$
(that is, S is a totally ordered subset of D).
- $(D, \sqsubseteq)$ has finite height if all chains are finite. In this case, its height is $\max\{|S| \mid S \text{ chain in } D\} - 1$.

Example

1. Every $S \subseteq \mathbb{N}$ is a chain in $(\mathbb{N}, \leq)$ (which is of infinite height)
2. $\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots\}$ is a chain in $(2^{\mathbb{N}}, \subseteq)$
3. $\{\emptyset, \{0\}, \{1\}\}$ is not a chain in $(2^{\mathbb{N}}, \subseteq)$

The Ascending Chain Condition

Termination of fixpoint iteration is guaranteed by the following condition.

Definition (Ascending Chain Condition)

- A sequence $(d_i)_{i \in \mathbb{N}}$ is called an ascending chain in D if $d_i \sqsubseteq d_{i+1}$ for each $i \in \mathbb{N}$.
- A partial order $(D, \sqsubseteq)$ satisfies the Ascending Chain Condition (ACC) if each ascending chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ eventually stabilizes, i.e., there exists $n \in \mathbb{N}$ such that $d_n = d_{n+1} = \ldots$

Notes:

- The finite height property implies ACC, but not vice versa (as there might be non-stabilizing descending chains)
- The complete lattice and ACC properties are orthogonal

Outline

Monotonicity of Functions

The monotonicity of transfer functions excludes "oscillating behavior" in fixpoint iteration.

Definition 4.1 (Monotonicity)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be partial orders, and let $\Phi : D \to D'$. Φ is called monotonic (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every $d_1, d_2 \in D$,

$$d_1 \sqsubseteq d_2 \implies \Phi(d_1) \sqsubseteq' \Phi(d_2).$$

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Example 4.2

1. Let $T := \{S \subseteq \mathbb{N} \mid S \text{ finite}\}$. Then $\Phi_1 : T \to \mathbb{N} : S \mapsto \sum_{n \in S} n$ is monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$ and $(\mathbb{N}, \leq)$.

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2. $\Phi_2 : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : S \mapsto \mathbb{N} \setminus S$ is not monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$ (since, e.g., $\emptyset \subseteq \mathbb{N}$ but $\Phi_2(\emptyset) = \mathbb{N} \not\subseteq \Phi_2(\mathbb{N}) = \emptyset$).

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3. (Live Variables) $(D, \sqsubseteq) = (D', \sqsubseteq') = (2^{Var_c}, \subseteq)$
   Each transfer function $\varphi_{l'}(V) := (V \setminus \text{kill}_{\text{LV}}(B^{l'})) \cup \text{gen}_{\text{LV}}(B^{l'})$ is obviously monotonic

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   Each transfer function $\varphi_{l'}(V) := (V \setminus \mathrm{kill}_{\mathrm{LV}}(B^{l'})) \cup \mathrm{gen}_{\mathrm{LV}}(B^{l'})$ is obviously monotonic
4. (Available Expressions) $(D, \sqsubseteq) = (D', \sqsubseteq') = (2^{CExp_c}, \supseteq)$ ditto

Fixpoints

Definition 4.3 (Fixpoint)

Let D be some domain, $d \in D$, and $\Phi : D \to D$. If

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Example 4.4

The (only) fixpoints of $\Phi : \mathbb{N} \to \mathbb{N} : n \mapsto n^2$ are 0 and 1

The Fixpoint Theorem I

Alfred Tarski (1901–1983)

Bronislaw Knaster (1893–1990)

Theorem 4.5 (Fixpoint Theorem by Tarski and Knaster)

Let $(D, \sqsubseteq)$ be a complete lattice satisfying ACC and $\Phi : D \to D$ monotonic. Then

$$\mathrm{fix}(\Phi) := \bigsqcup \left\{ \Phi^k(\bot) \mid k \in \mathbb{N} \right\}$$

is the least fixpoint of Φ where

$$\Phi^0(d) := d \text{ and } \Phi^{k+1}(d) := \Phi(\Phi^k(d)).$$

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Function requirements for dataflow analysis

All transfer functions must be a monotonic

The Fixpoint Theorem II

The proof of Theorem 4.5 requires the following lemma.

Lemma 4.6

Let $(D, \sqsubseteq)$ *be a complete lattice satisfying ACC,* $S \subseteq D$ *a chain, and* $\Phi : D \to D$ *monotonic. Then*

$$\Phi\left(\bigsqcup S\right) = \bigsqcup \Phi(S)$$

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on the board □

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on the board □

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on the board □

Remark: ACC $\implies$ $\left(\Phi^k(\bot)\right)_{k \in \mathbb{N}}$ stabilizes at some $k_0 \in \mathbb{N}$ with $\mathrm{fix}(\Phi) = \Phi^{k_0}(\bot)$ (where k_0 bounded by height of $(D, \sqsubseteq)$)

Outline

Dataflow Systems I

Definition 4.7 (Dataflow system)

A dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ consists of

- a finite set of (program) labels Lab (here: Lab_c),
- a set of extremal labels $E \subseteq Lab$ (here: $\{\mathsf{init}(c)\}$ or $\mathsf{final}(c)$),
- a flow relation $F \subseteq Lab \times Lab$ (here: $\mathsf{flow}(c)$ or $\mathsf{flow}^R(c)$),
- a complete lattice $(D, \sqsubseteq)$ satisfying ACC
  (with LUB operator $\bigsqcup$ and least element $\bot$),
- an extremal value $\iota \in D$ (for the extremal labels), and
- a collection of monotonic transfer functions $\{\varphi_l \mid l \in Lab\}$ of type $\varphi_l : D \to D$.

Example 4.8

Problem	Available Expressions	Live Variables
E	$\{\mathrm{init}(c)\}$	$\mathrm{final}(c)$
F	$\mathrm{flow}(c)$	$\mathrm{flow}^R(c)$
D	2^{CExp_c}	2^{Var_c}
$\sqsubseteq$	$\supseteq$	$\subseteq$
$\bigsqcup$	$\bigcap$	$\bigcup$
$\bot$	$CExp_c$	$\emptyset$
ι	$\emptyset$	Var_c
φ_l	$\varphi_l(d) = (d \setminus \mathrm{kill}(B^l)) \cup \mathrm{gen}(B^l)$	

Dataflow Systems and Fixpoints

Definition 4.9 (Dataflow equation system)

Given: dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$, $Lab = \{1, \ldots, n\}$ (w.l.o.g.)

- S determines the equation system (where $l \in Lab$)

$$\mathsf{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

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- $(d_1, \ldots, d_n) \in D^n$ is called a solution if

$$d_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(d_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

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- S determines the transformation

$$\Phi_S : D^n \to D^n : (d_1, \ldots, d_n) \mapsto (d'_1, \ldots, d'_n)$$

where

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Corollary 4.10

$(d_1, \ldots, d_n) \in D^n$ solves the equation system iff it is a fixpoint of Φ_S

Solving Dataflow Problems by Fixpoint Iteration

Remarks:

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- Thus the (least) fixpoint is effectively computable by iteration:

$$\mathrm{fix}(\Phi_S) = \bigsqcup \{\Phi_S^k(\bot_{D^n}) \mid k \in \mathbb{N}\}$$

where $\bot_{D^n} = (\underbrace{\bot_D, \ldots, \bot_D}_{n \text{ times}})$

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- If height of $(D, \sqsubseteq)$ is m
  $\Longrightarrow$ height of $(D^n, \sqsubseteq^n)$ is $m \cdot n$
  $\Longrightarrow$ fixpoint iteration requires at most $m \cdot n$ steps

Example: Available Expressions

Example 4.11 (Available Expressions; cf. Example 2.9)

Program:

$c =$ [x := a+b]^1;
    [y := a*b]^2;
    while [y > a+b]^3 do
      [a := a+1]^4;
      [x := a+b]^5

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Equation system:

$$
\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\}) \\
\mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}
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Fixpoint iteration:

i	1	2	3	4	5
0	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$

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\begin{aligned}
AE_1 &= \emptyset\\
AE_2 &= AE_1 \cup \{\texttt{a+b}\}\\
AE_3 &= (AE_2 \cup \{\texttt{a*b}\}) \cap (AE_5 \cup \{\texttt{a+b}\})\\
AE_4 &= AE_3 \cup \{\texttt{a+b}\}\\
AE_5 &= AE_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5
0	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$
1	$\emptyset$	$CExp_c$	$CExp_c$	$CExp_c$	$\emptyset$

Example: Available Expressions

Example 4.11 (Available Expressions; cf. Example 2.9)

Program:

$c = [\texttt{x := a+b}]^1;$
$\quad [\texttt{y := a*b}]^2;$
$\quad \texttt{while } [\texttt{y > a+b}]^3 \texttt{ do}$
$\qquad [\texttt{a := a+1}]^4;$
$\qquad [\texttt{x := a+b}]^5$

Equation system:

$$
\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\}) \\
\mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5
0	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$
1	$\emptyset$	$CExp_c$	$CExp_c$	$CExp_c$	$\emptyset$
2	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$CExp_c$	$\emptyset$

Example: Available Expressions

Example 4.11 (Available Expressions; cf. Example 2.9)

Program:

$$
\begin{aligned}
c = {} & [\mathtt{x\ :=\ a+b}]^1;\\
& [\mathtt{y\ :=\ a*b}]^2;\\
& \mathtt{while}\ [\mathtt{y\ >\ a+b}]^3\ \mathtt{do}\\
& \quad [\mathtt{a\ :=\ a+1}]^4;\\
& \quad [\mathtt{x\ :=\ a+b}]^5
\end{aligned}
$$

Equation system:

$$
\begin{aligned}
\mathsf{AE}_1 &= \emptyset\\
\mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\mathtt{a+b}\}\\
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\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5
0	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$
1	$\emptyset$	$CExp_c$	$CExp_c$	$CExp_c$	$\emptyset$
2	$\emptyset$	$\{\mathtt{a+b}\}$	$\{\mathtt{a+b}\}$	$CExp_c$	$\emptyset$
3	$\emptyset$	$\{\mathtt{a+b}\}$	$\{\mathtt{a+b}\}$	$\{\mathtt{a+b}\}$	$\emptyset$

Example: Available Expressions

Example 4.11 (Available Expressions; cf. Example 2.9)

Program:

$$
\begin{array}{l}
c = [\texttt{x := a+b}]^1;\\
\quad [\texttt{y := a*b}]^2;\\
\quad \texttt{while } [\texttt{y > a+b}]^3 \texttt{ do}\\
\qquad [\texttt{a := a+1}]^4;\\
\qquad [\texttt{x := a+b}]^5
\end{array}
$$

Equation system:

$$
\begin{aligned}
AE_1 &= \emptyset\\
AE_2 &= AE_1 \cup \{\texttt{a+b}\}\\
AE_3 &= (AE_2 \cup \{\texttt{a*b}\}) \cap (AE_5 \cup \{\texttt{a+b}\})\\
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\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5
0	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$	$CExp_c$
1	$\emptyset$	$CExp_c$	$CExp_c$	$CExp_c$	$\emptyset$
2	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$CExp_c$	$\emptyset$
3	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\emptyset$
4	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\emptyset$

Example: Live Variables

Example 4.12 (Live Variables; cf. Example 2.12)

Program:

```
[x := 2]^1;[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
  [z := x]^5
else
  [z := y*y]^6;
[x := z]^7
```

Example: Live Variables

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```
[x := 2]^1;[y := 4]^2;
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if [y > 0]^4 then
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else
  [z := y*y]^6;
[x := z]^7
```

Equation system:

$$
\begin{aligned}
\mathsf{LV}_1 &= \mathsf{LV}_2 \setminus \{y\} \\
\mathsf{LV}_2 &= \mathsf{LV}_3 \setminus \{x\} \\
\mathsf{LV}_3 &= \mathsf{LV}_4 \cup \{y\} \\
\mathsf{LV}_4 &= ((\mathsf{LV}_5 \setminus \{z\}) \cup \{x\}) \cup ((\mathsf{LV}_6 \setminus \{z\}) \cup \{y\}) \\
\mathsf{LV}_5 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\} \\
\mathsf{LV}_6 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\} \\
\mathsf{LV}_7 &= \{x, y, z\}
\end{aligned}
$$

Example: Live Variables

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Program:

```
[x := 2]^1;[y := 4]^2;
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\mathsf{LV}_6 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\} \\
\mathsf{LV}_7 &= \{x, y, z\}
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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Example 4.12 (Live Variables; cf. Example 2.12)

Program:

```
[x := 2]^1;[y := 4]^2;
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\mathsf{LV}_6 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}\\
\mathsf{LV}_7 &= \{x, y, z\}
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\{x, y, z\}$

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Program:

```
[x := 2]^1;[y := 4]^2;
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if [y > 0]^4 then
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LV_6 &= (LV_7 \setminus \{x\}) \cup \{z\} \\
LV_7 &= \{x, y, z\}
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\{x, y, z\}$
2	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{y, z\}$	$\{y, z\}$	$\{x, y, z\}$

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Example 4.12 (Live Variables; cf. Example 2.12)

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```
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\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\{x, y, z\}$
2	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{y, z\}$	$\{y, z\}$	$\{x, y, z\}$
3	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{y, z\}$	$\{y, z\}$	$\{x, y, z\}$