Static Program Analysis

Lecture 2: Dataflow Analysis I
(Introduction & Available Expressions/Live Variables Analysis)

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Outline

1. Preliminaries on Dataflow Analysis
2. An Example: Available Expressions Analysis
3. Another Example: Live Variables Analysis

Dataflow Analysis: the Approach

- Traditional form of program analysis
- Idea: describe how analysis information flows through program
- Distinctions:

dependence on statement order:
: flow-sensitive vs. flow-insensitive analyses

direction of flow:
: forward vs. backward analyses

quantification over paths:
: may (union) vs. must (intersection) analyses

procedures:
: interprocedural vs. intraprocedural analyses

Labelled Programs

- Goal: localisation of analysis information
- Dataflow information will be associated with
  - `skip` statements
  - assignments
  - tests in conditionals (`if`) and loops (`while`)
- Assume set of labels Lab with meta variable $l \in Lab$ (usually $Lab = \mathbb{N}$)

Definition 2.1 (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context-free grammar:

$$\begin{aligned}
a &::= z \mid x \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 * a_2 \in AExp \\
b &::= t \mid a_1 = a_2 \mid a_1 > a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\
c &::= [\texttt{skip}]^l \mid [x \texttt{ := } a]^l \mid c_1 ; c_2 \mid \\
&\qquad \texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } [b]^l \texttt{ do } c \in Cmd
\end{aligned}$$

- All labels in $c \in Cmd$ assumed distinct, denoted by Lab_c
- Labelled fragments of c called blocks, denoted by Blk_c

A WHILE Program with Labels

Example 2.2

```
x := 6;
y := 7;
z := 0;
while x > 0 do
  x := x - 1;
  v := y;
  while v > 0 do
    v := v - 1;
    z := z + 1
```

Representing Control Flow I

Every (labelled) statement has a single entry (given by the initial label) and generally multiple exits (given by the final labels):

Definition 2.3 (Initial and final labels)

The mapping $\mathsf{init} : \mathit{Cmd} \to \mathit{Lab}$ returns the initial label of a statement:

$$\begin{aligned}
\mathsf{init}([\mathtt{skip}]^l) &:= l \\
\mathsf{init}([x \mathrel{\mathtt{:=}} a]^l) &:= l \\
\mathsf{init}(c_1 \,;c_2) &:= \mathsf{init}(c_1) \\
\mathsf{init}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= l \\
\mathsf{init}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= l
\end{aligned}$$

The mapping $\mathsf{final} : \mathit{Cmd} \to 2^{\mathit{Lab}}$ returns the set of final labels of a statement:

$$\begin{aligned}
\mathsf{final}([\mathtt{skip}]^l) &:= \{l\} \\
\mathsf{final}([x \mathrel{\mathtt{:=}} a]^l) &:= \{l\} \\
\mathsf{final}(c_1 \,;c_2) &:= \mathsf{final}(c_2) \\
\mathsf{final}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= \mathsf{final}(c_1) \cup \mathsf{final}(c_2) \\
\mathsf{final}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= \{l\}
\end{aligned}$$

Representing Control Flow II

Definition 2.4 (Flow relation)

Given a statement $c \in Cmd$, the (control) flow relation

$$\mathrm{flow}(c) \subseteq Lab \times Lab$$

is defined by

$$\begin{aligned}
\mathrm{flow}([\mathtt{skip}]^l) &:= \emptyset \\
\mathrm{flow}([x \mathrel{\mathtt{:=}} a]^l) &:= \emptyset \\
\mathrm{flow}(c_1 ; c_2) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \\
&\ \{(l, \mathrm{init}(c_2)) \mid l \in \mathrm{final}(c_1)\} \\
\mathrm{flow}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \\
&\ \{(l, \mathrm{init}(c_1)), (l, \mathrm{init}(c_2))\} \\
\mathrm{flow}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= \mathrm{flow}(c) \cup \{(l, \mathrm{init}(c))\} \cup \\
&\ \{(l', l) \mid l' \in \mathrm{final}(c)\}
\end{aligned}$$

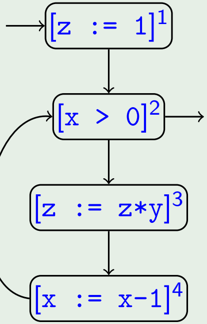
Representing Control Flow III

Example 2.5

$$\begin{aligned} c = {} & [\texttt{z := 1}]^1; \\ & \quad \texttt{while } [\texttt{x > 0}]^2 \texttt{ do} \\ & \qquad [\texttt{z := z*y}]^3; \\ & \qquad [\texttt{x := x-1}]^4 \end{aligned}$$

$$\begin{aligned} \mathsf{init}(c) &= 1 \\ \mathsf{final}(c) &= \{2\} \\ \mathsf{flow}(c) &= \{(1,2),(2,3),(3,4),(4,2)\} \end{aligned}$$

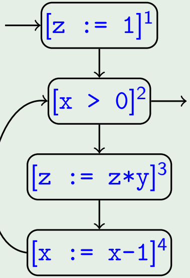
Visualization by (control) flow graph:



Representing Control Flow IV

- To simplify the presentation we will often assume that the program $c \in Cmd$ under consideration has an isolated entry, meaning that
$$\{l \in Lab \mid (l, \mathrm{init}(c)) \in \mathrm{flow}(c)\} = \emptyset$$
(which is the case when c does not start with a `while` loop)
- Similarly: $c \in Cmd$ has isolated exits if
$$\{l' \in Lab \mid (l, l') \in \mathrm{flow}(c) \text{ for some } l \in \mathrm{final}(c)\} = \emptyset$$
(which is the case when no final label identifies a loop header)

Example 2.6 (cf. Example 2.5)



has an isolated entry but not isolated exits

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2. An Example: Available Expressions Analysis
3. Another Example: Live Variables Analysis

Goal of Available Expressions Analysis

Available Expressions Analysis

The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

- Can be used for Common Subexpression Elimination: replace subexpression by variable that contains up-to-date value
- Only interesting for non-trivial (i.e., complex) arithmetic expressions

Example 2.7 (Available Expressions Analysis)

```
[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
```

- a+b available at label 3
- a+b not available at label 5
- possible optimization:
  `while [y > x]^3 do`

Formalizing Available Expressions Analysis I

- Given $a \in AExp$, $b \in BExp$, $c \in Cmd$
  - $Var_a/Var_b/Var_c$ denotes the set of all variables occurring in $a/b/c$
  - $CExp_b/CExp_c$ denote the sets of all complex arithmetic expressions occurring in b/c
- An expression a is killed in a block B if any of the variables in a is modified in B
- Formally: $\mathrm{kill}_{\mathrm{AE}} : Blk_c \to 2^{CExp_c}$ is defined by

$$\begin{aligned}
\mathrm{kill}_{\mathrm{AE}}([\mathtt{skip}]^l) &:= \emptyset \\
\mathrm{kill}_{\mathrm{AE}}([x\ \mathtt{:=}\ a]^l) &:= \{a' \in CExp_c \mid x \in Var_{a'}\} \\
\mathrm{kill}_{\mathrm{AE}}([b]^l) &:= \emptyset
\end{aligned}$$

- An expression a is generated in a block B if it is evaluated in and none of its variables are modified by B
- Formally: $\mathrm{gen}_{\mathrm{AE}} : Blk_c \to 2^{CExp_c}$ is defined by

$$\begin{aligned}
\mathrm{gen}_{\mathrm{AE}}([\mathtt{skip}]^l) &:= \emptyset \\
\mathrm{gen}_{\mathrm{AE}}([x\ \mathtt{:=}\ a]^l) &:= \{a \mid x \notin Var_a\} \\
\mathrm{gen}_{\mathrm{AE}}([b]^l) &:= CExp_b
\end{aligned}$$

Formalizing Available Expressions Analysis II

Example 2.8 ($\mathsf{kill}_{\mathsf{AE}}/\mathsf{gen}_{\mathsf{AE}}$ functions)

```
c = [x := a+b]^1;
    [y := a*b]^2;
    while [y > a+b]^3 do
      [a := a+1]^4;
      [x := a+b]^5
```

- $CExp_c = \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}$
-

Lab_c	$\mathsf{kill}_{\mathsf{AE}}(B^l)$	$\mathsf{gen}_{\mathsf{AE}}(B^l)$
1	$\emptyset$	$\{\mathtt{a+b}\}$
2	$\emptyset$	$\{\mathtt{a*b}\}$
3	$\emptyset$	$\{\mathtt{a+b}\}$
4	$\{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}$	$\emptyset$
5	$\emptyset$	$\{\mathtt{a+b}\}$

The Equation System I

- Analysis itself defined by setting up an equation system
- For each $l \in Lab_c$, $\mathrm{AE}_l \subseteq CExp_c$ represents the set of available expressions at the entry of block B^l
- Formally, for $c \in Cmd$ with isolated entry:

$$\mathrm{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathrm{init}(c) \\ \bigcap\{\varphi_{l'}(\mathrm{AE}_{l'}) \mid (l', l) \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$

  where $\varphi_{l'} : 2^{CExp_c} \to 2^{CExp_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(A) := (A \setminus \mathrm{kill}_{\mathrm{AE}}(B^{l'})) \cup \mathrm{gen}_{\mathrm{AE}}(B^{l'})$$

- Characterization of analysis:

  flow-sensitive: results depending on order of assignments

  forward: starts in $\mathrm{init}(c)$ and proceeds downwards

  must: $\bigcap$ in equation for AE_l

- Later: solution not necessarily unique

  $\Longrightarrow$ choose greatest one

The Equation System II

Reminder:

$$\mathsf{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathsf{init}(c) \\ \bigcap\{\varphi_{l'}(\mathsf{AE}_{l'}) \mid (l', l) \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(E) = (E \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

Example 2.9 (AE equation system)

$c =$
```
[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
```

$l \in Lab_c$	$\mathsf{kill}_{\mathsf{AE}}(B^l)$	$\mathsf{gen}_{\mathsf{AE}}(B^l)$
1	$\emptyset$	{a+b}
2	$\emptyset$	{a*b}
3	$\emptyset$	{a+b}
4	{a+b, a*b, a+1}	$\emptyset$
5	$\emptyset$	{a+b}

Equations:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \varphi_1(\mathsf{AE}_1) = \mathsf{AE}_1 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\
&= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\}) \\
\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}$$

Solution:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= \{\texttt{a+b}\} \\
\mathsf{AE}_4 &= \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \emptyset
\end{aligned}$$

Outline

1 Preliminaries on Dataflow Analysis

2 An Example: Available Expressions Analysis

3 Another Example: Live Variables Analysis

Goal of Live Variables Analysis

Live Variables Analysis

The goal of Live Variables Analysis is to determine, for each program point, which variables *may* be live at the exit from the point.

- A variable is called live at the exit from a block if there exists a path from the block to a use of the variable that does not re-define the variable
- All variables considered to be live at the end of the program (alternative: restriction to output variables)
- Can be used for Dead Code Elimination:
  remove assignments to non-live variables

An Example

Example 2.10 (Live Variables Analysis)

```
[x := 2]^1;
[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
  [z := x]^5
else
  [z := y*y]^6;
[x := z]^7
```

- x not live at exit from label 1
- y live at exit from 2
- x live at exit from 3
- z live at exits from 5 and 6
- possible optimization: remove $[\texttt{x := 2}]^1$

Formalizing Live Variables Analysis I

- A variable on the left-hand side of an assignment is killed by the assignment; tests and `skip` do not kill
- Formally: $\mathsf{kill}_{\mathsf{LV}} : Blk_c \to 2^{Var_c}$ is defined by

$$\begin{aligned}
\mathsf{kill}_{\mathsf{LV}}([\mathtt{skip}]^l) &:= \emptyset \\
\mathsf{kill}_{\mathsf{LV}}([x \mathtt{\ :=\ } a]^l) &:= \{x\} \\
\mathsf{kill}_{\mathsf{LV}}([b]^l) &:= \emptyset
\end{aligned}$$

- Every reading access generates a live variable
- Formally: $\mathsf{gen}_{\mathsf{LV}} : Blk_c \to 2^{Var_c}$ is defined by

$$\begin{aligned}
\mathsf{gen}_{\mathsf{LV}}([\mathtt{skip}]^l) &:= \emptyset \\
\mathsf{gen}_{\mathsf{LV}}([x \mathtt{\ :=\ } a]^l) &:= Var_a \\
\mathsf{gen}_{\mathsf{LV}}([b]^l) &:= Var_b
\end{aligned}$$

Example 2.11 ($\mathrm{kill}_{\mathrm{LV}}/\mathrm{gen}_{\mathrm{LV}}$ functions)

c =
```
[x := 2]^1;
[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
  [z := x]^5
else
  [z := y*y]^6;
[x := z]^7
```

- $Var_c = \{x, y, z\}$
-

$l \in Lab_c$	$\mathrm{kill}_{\mathrm{LV}}(B^l)$	$\mathrm{gen}_{\mathrm{LV}}(B^l)$
1	$\{x\}$	$\emptyset$
2	$\{y\}$	$\emptyset$
3	$\{x\}$	$\emptyset$
4	$\emptyset$	$\{y\}$
5	$\{z\}$	$\{x\}$
6	$\{z\}$	$\{y\}$
7	$\{x\}$	$\{z\}$

The Equation System I

- For each $l \in Lab_c$, $\mathsf{LV}_l \subseteq Var_c$ represents the set of live variables at the exit of block B^l
- Formally, for a program $c \in Cmd$ with isolated exits:
$$\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l, l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$
where $\varphi_{l'} : 2^{Var_c} \to 2^{Var_c}$ denotes the transfer function of block $B^{l'}$, given by
$$\varphi_{l'}(V) := (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$$
- Characterization of analysis:

  flow-sensitive: results depending on order of assignments

  backward: starts in $\mathsf{final}(c)$ and proceeds upwards

  may: $\bigcup$ in equation for LV_l
- Later: solution not necessarily unique

  $\Longrightarrow$ choose least one

The Equation System II

Reminder:

$$\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l,l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(V) = (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$$

Example 2.12 (LV equation system)

$$
\begin{aligned}
c = {} & [\mathtt{x\ :=\ 2}]^1; [\mathtt{y\ :=\ 4}]^2;\\
& [\mathtt{x\ :=\ 1}]^3;\\
& \mathtt{if}\ [\mathtt{y\ >\ 0}]^4\ \mathtt{then}\\
& \quad [\mathtt{z\ :=\ x}]^5\\
& \mathtt{else}\\
& \quad [\mathtt{z\ :=\ y*y}]^6;\\
& [\mathtt{x\ :=\ z}]^7
\end{aligned}
$$

$l \in Lab_c$	$\mathsf{kill}_{\mathsf{LV}}(B^l)$	$\mathsf{gen}_{\mathsf{LV}}(B^l)$
1	$\{\mathtt{x}\}$	$\emptyset$
2	$\{\mathtt{y}\}$	$\emptyset$
3	$\{\mathtt{x}\}$	$\emptyset$
4	$\emptyset$	$\{\mathtt{y}\}$
5	$\{\mathtt{z}\}$	$\{\mathtt{x}\}$
6	$\{\mathtt{z}\}$	$\{\mathtt{y}\}$
7	$\{\mathtt{x}\}$	$\{\mathtt{z}\}$

$$
\begin{aligned}
\mathsf{LV}_1 &= \varphi_2(\mathsf{LV}_2) = \mathsf{LV}_2 \setminus \{\mathtt{y}\}\\
\mathsf{LV}_2 &= \varphi_3(\mathsf{LV}_3) = \mathsf{LV}_3 \setminus \{\mathtt{x}\}\\
\mathsf{LV}_3 &= \varphi_4(\mathsf{LV}_4) = \mathsf{LV}_4 \cup \{\mathtt{y}\}\\
\mathsf{LV}_4 &= \varphi_5(\mathsf{LV}_5) \cup \varphi_6(\mathsf{LV}_6)\\
&= ((\mathsf{LV}_5 \setminus \{\mathtt{z}\}) \cup \{\mathtt{x}\}) \cup ((\mathsf{LV}_6 \setminus \{\mathtt{z}\}) \cup \{\mathtt{y}\})\\
\mathsf{LV}_5 &= \varphi_7(\mathsf{LV}_7) = (\mathsf{LV}_7 \setminus \{\mathtt{x}\}) \cup \{\mathtt{z}\}\\
\mathsf{LV}_6 &= \varphi_7(\mathsf{LV}_7) = (\mathsf{LV}_7 \setminus \{\mathtt{x}\}) \cup \{\mathtt{z}\}\\
\mathsf{LV}_7 &= \{\mathtt{x},\mathtt{y},\mathtt{z}\}
\end{aligned}
$$

Solution:

$$
\begin{aligned}
\mathsf{LV}_1 &= \emptyset\\
\mathsf{LV}_2 &= \{\mathtt{y}\}\\
\mathsf{LV}_3 &= \{\mathtt{x},\mathtt{y}\}\\
\mathsf{LV}_4 &= \{\mathtt{x},\mathtt{y}\}\\
\mathsf{LV}_5 &= \{\mathtt{y},\mathtt{z}\}\\
\mathsf{LV}_6 &= \{\mathtt{y},\mathtt{z}\}\\
\mathsf{LV}_7 &= \{\mathtt{x},\mathtt{y},\mathtt{z}\}
\end{aligned}
$$