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Exercise 1 (Operational Semantics of WHILE): (2 Points)

Prove that the execution relation as presented in the lecture is deterministic. (Note that a (mathematical) function is deterministic by definition.)

Exercise 2 (Abstract Transition System): (4 Points)

For a single integer, modulo abstraction is defined by the mapping $\mathbb{Z} \to \{0, \ldots, n-1\} : z \mapsto (z \mod n)$ for some fixed $n \geq 1$.

a) Give the definition of the corresponding abstraction and concretization functions operating on sets of integers, and show that they form a Galois connection.

b) Extract the functions $+^\sharp_n$, $*^\sharp_n$, $(\mod m)^\sharp_n$ and relations $=^\sharp_n$, $>^\sharp_n$ as safe approximations of $+$, $*$, $\mod m$, $=$ and $>$.

c) Depict the reachable fragment of the abstract transition system for the following `WHILE`-program for the modulo abstraction with $n = 4$.

```
x := 3 * x;
while (¬(x mod 4 = 0))
   if (x mod 4 = 1)
      x := 3 * x;
   x := x + 1;
```

Exercise 3 (Galois Insertions): (4 Points)

In every Galois connection we considered so far we observed the special case that $\alpha(\gamma(m)) = m$. These Galois connections are referred to as *Galois insertions*:

(α, γ) is a *Galois insertion* between the complete lattices L and M if and only if:

$$\alpha : L \to M \textit{ and } \gamma : M \to L \text{ are monotone functions}$$

that satisfy:

$$\begin{array}{rcll} \gamma(\alpha(l)) & \sqsupseteq & l & \forall\, l \in L \\ \alpha(\gamma(m)) & = & m & \forall\, m \in M \end{array}$$

Show that for a Galois connection (α, γ) between L and M the following claims are equivalent:

(i) (α, γ) is a Galois insertion

(ii) γ is injective

(iii) α is surjective

(iv) $\forall\, m_1, m_2 : \; m_1 \sqsubseteq m_2 \Leftrightarrow \gamma(m_1) \sqsubseteq \gamma(m_2)$