# Theoretical Foundations of the UML

## Lecture 9: Bounded MSC and CFMs


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http://moves.rwth-aachen.de/teaching/ws-1415/uml/

17. November 2014

# Outline

1. Communicating finite-state machines: a refresher
2. Well-formedness of CFMs
3. Bounded CFMs
   - Bounded words
   - Bounded MSCs
   - Bounded CFMs

# Overview

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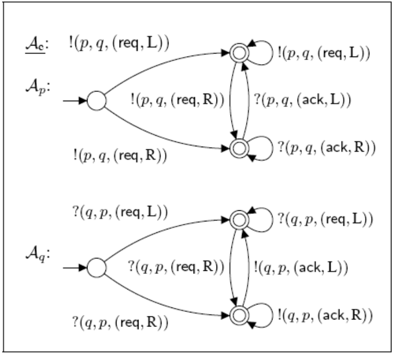
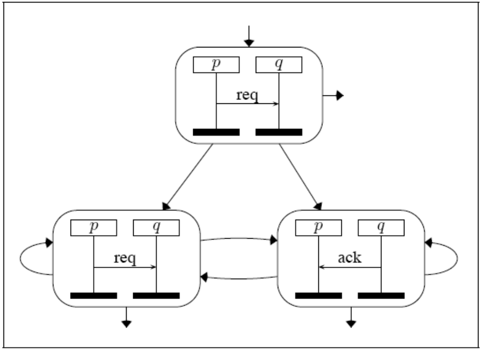
# Communicating finite-state machines

- A communicating finite-state machine (CFM) is a collection of finite-state machines, one for each process
- Communication between these machines takes place via (a priori) unbounded reliable FIFO channels
- The underlying system architecture is parametrised by the set $\mathcal{P}$ of processes and the set $\mathcal{C}$ of messages
- Action $!(p, q, m)$ puts message $m$ at the end of the channel $(p, q)$
- Action $?(q, p, m)$ is enabled only if $m$ is at head of buffer, and its execution by process $q$ removes $m$ from the channel $(p, q)$
- Synchronisation messages are used to avoid deadlocks

# Example communicating finite-state machine




This CFM accepts if $\mathcal{A}_p$ and $\mathcal{A}_q$ are in some local state, and (as usual) all channels are empty

# Formal definition

## Definition (What is a CFM?)

A communicating finite-state machine (CFM) over $\mathcal{P}$ and $\mathcal{C}$ is a tuple

$$\mathcal{A} = (((S_p, \Delta_p))_{p \in \mathcal{P}}, \mathbb{D}, s_{init}, F)$$

where

- for each $p \in \mathcal{P}$:
  - $S_p$ is a non-empty finite set of local states (the $S_p$ are disjoint)
  - $\Delta_p \subseteq S_p \times Act_p \times \mathbb{D} \times S_p$ is a set of local transitions
- $\mathbb{D}$ is a nonempty finite set of synchronization messages (or data)
- $s_{init} \in S_{\mathcal{A}}$ is the global initial state
  - where $S_{\mathcal{A}} := \prod_{p \in \mathcal{P}} S_p$ is the set of global states of $\mathcal{A}$
- $F \subseteq S_{\mathcal{A}}$ is the set of global final states

In sequel, let $\mathcal{A} = (((S_p, \Delta_p))_{p \in \mathcal{P}}, \mathbb{D}, s_{init}, F)$ be a CFM over $\mathcal{P}$ and $\mathcal{C}$.

# Formal semantics of CFMs

## Definition (Configuration)

Configurations of $\mathcal{A}$: $\mathit{Conf}_{\mathcal{A}} := S_{\mathcal{A}} \times \{\eta \mid \eta : \mathit{Ch} \to (\mathcal{C} \times \mathbb{D})^*\}$

## Definition (Transitions between configurations)

$\Longrightarrow_{\mathcal{A}} \subseteq \mathit{Conf}_{\mathcal{A}} \times \mathit{Act} \times \mathbb{D} \times \mathit{Conf}_{\mathcal{A}}$ is defined as follows:

- sending a message: $((\overline{s}, \eta), !(p, q, a), m, (\overline{s}', \eta')) \in \Longrightarrow_{\mathcal{A}}$ if
  - $(\overline{s}[p], !(p, q, a), m, \overline{s}'[p]) \in \Delta_p$
  - $\eta' = \eta[(p, q) := (a, m) \cdot \eta((p, q))]$
  - $\overline{s}[r] = \overline{s}'[r]$ for all $r \in \mathcal{P} \setminus \{p\}$
- receipt of a message: $((\overline{s}, \eta), ?(p, q, a), m, (\overline{s}', \eta')) \in \Longrightarrow_{\mathcal{A}}$ if
  - $(\overline{s}[p], ?(p, q, a), m, \overline{s}'[p]) \in \Delta_p$
  - $\eta((q, p)) = w \cdot (a, m) \neq \epsilon$ and $\eta' = \eta[(q, p) := w]$
  - $\overline{s}[r] = \overline{s}'[r]$ for all $r \in \mathcal{P} \setminus \{p\}$

# Linearizations of a CFM

## Definition ((Accepting) Runs)

A run of $\mathcal{A}$ on $\sigma_1 \ldots \sigma_n \in Act^*$ is a sequence $\rho = \gamma_0 \, m_1 \, \gamma_1 \ldots \gamma_{n-1} \, m_n \, \gamma_n$ such that

- $\gamma_0 = (s_{init}, \eta_\varepsilon)$ with $\eta_\varepsilon$ mapping any channel to $\varepsilon$
- $\gamma_{i-1} \xRightarrow{\sigma_i, m_i}_{\mathcal{A}} \gamma_i$ for any $i \in \{1, \ldots, n\}$

Run $\rho$ is accepting if $\gamma_n \in F \times \{\eta_\varepsilon\}$.

## Definition (Linearizations)

The set of linearizations of CFM $\mathcal{A}$:

$Lin(\mathcal{A}) := \{w \in Act^* \mid \text{there is an accepting run of } \mathcal{A} \text{ on } w\}$

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# Well-formedness (reminder)

Let $Ch := \{(p,q) \mid p \neq q\,,\ p,q \in \mathcal{P}\}$ be a set of channels over $\mathcal{P}$.

We call $w = a_1 \ldots a_n \in Act^*$ proper if

1. every receive in $w$ is preceded by a corresponding send, i.e.:
   $\forall (p,q) \in Ch$ and prefix $u$ of $w$, we have:

$$\underbrace{\sum_{m \in \mathcal{C}} |u|_{!(p,q,m)}}_{\#\text{ sends from } p \text{ to } q} \quad \geqslant \quad \underbrace{\sum_{m \in \mathcal{C}} |u|_{?(q,p,m)}}_{\#\text{ receipts by } q \text{ from } p}$$

   where $|u|_a$ denotes the number of occurrences of action $a$ in $u$

2. the FIFO policy is respected, i.e.:
   $\forall 1 \leqslant i < j \leqslant n$, $(p,q) \in Ch$, and $a_i = !(p,q,m_1)$, $a_j = ?(q,p,m_2)$:

$$\sum_{m \in \mathcal{C}} |a_1 \ldots a_{i-1}|_{!(p,q,m)} \;=\; \sum_{m \in \mathcal{C}} |a_1 \ldots a_{j-1}|_{?(q,p,m)} \quad \text{implies} \quad m_1 = m_2$$

A proper word $w$ is well-formed if $\sum_{m \in \mathcal{C}} |w|_{!(p,q,m)} = \sum_{m \in \mathcal{C}} |w|_{?(q,p,m)}$

# Well-formedness and CFMs

**Lemma**

For any CFM $\mathcal{A}$ and $w \in Lin(\mathcal{A})$, $w$ is well-formed.

Recall that there is a strong correspondence between well-formed linearizations and MSCs.

# From linearizations to partial orders (reminder)

Associate to $w = a_1 \ldots a_n \in Act^*$ an $Act$-labelled poset

$$M(w) = (E, \preceq, \ell)$$

such that:

- $E = \{1, \ldots, n\}$ are the positions in $w$ labelled with $\ell(i) = a_i$
- $\preceq = \left( \prec_{\text{msg}} \cup \bigcup_{p \in \mathcal{P}} \prec_p \right)^*$ where
  - $i \prec_p j$ if and only if $i < j$ for any $i, j \in E_p$
  - $i \prec_{\text{msg}} j$ if for some $(p, q) \in Ch$ and $m \in \mathcal{C}$ we have:

$$\ell(i) = !(p, q, m) \text{ and } \ell(j) = ?(q, p, m) \text{ and}$$

$$\sum_{m \in \mathcal{C}} |a_1 \ldots a_{i-1}|_{!(p,q,m)} = \sum_{m \in \mathcal{C}} |a_1 \ldots a_{j-1}|_{?(q,p,m)}$$

# CFMs and well-formed words

### Relating well-formed words to MSCs

For any well-formed word $w \in Act^*$, $M(w)$ is an MSC.

### Definition (MSC language of a CFM)

For CFM $\mathcal{A}$, let $\mathcal{L}(\mathcal{A}) = \{ M(w) \mid w \in Lin(\mathcal{A}) \}$.

### Relating well-formed words to CFMs

For any well-formed words $u$ and $v$ with $M(u)$ is isomorphic to $M(v)$:

$$\text{for any CFM } \mathcal{A}: \; u \in \mathcal{L}(\mathcal{A}) \quad \text{iff} \quad v \in \mathcal{L}(\mathcal{A}).$$

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# Emptiness problem is undecidable for CFMs

**Theorem:** [Brand & Zafiropulo 1983]

The following (emptiness) problem:

| | |
|---|---|
| INPUT: | CFM $\mathcal{A}$ over processes $\mathcal{P}$ and message contents $\mathcal{C}$ |
| QUESTION: | Is $\mathcal{L}(\mathcal{A})$ empty? |

is undecidable. (Even if $\mathcal{C}$ is a singleton set).

# Restrictions on CFMs

- So: most elementary problems for CFMs are undecidable.
- This is (very) unsatisfactory.
- Main cause: presence of channels with unbounded capacity
- Consider restricted versions of CFMs by bounding the channel capacities.
- Thus: we fix the channel capacities a priori.
- This yields:
  - universally bounded CFMs: all runs need a finite buffer capacity
  - existentially bounded CFMs: some runs need a finite buffer capacity possibly, some runs still need unbounded buffers.

We define bounded CFMs, by first considering bounded words and bounded MSCs. Bounded CFMs will then generate bounded MSCs.

# Bounded words

## Definition ($B$-bounded words)

Let $B \in \mathbb{N}$ and $B > 0$. A word $w \in Act^*$ is called $B$-bounded if for any prefix $u$ of $w$ and any channel $(p, q) \in Ch$:

$$0 \leqslant \sum_{a \in \mathcal{C}} |u|_{!(p,q,a)} - \sum_{a \in \mathcal{C}} |u|_{?(q,p,a)} \leqslant B$$

## Intuition

Word $w$ is $B$-bounded if for any pair of processes $(p, q)$, the number of sends from $p$ to $q$ cannot be more than $B$ ahead of the number of receipts by $q$ from $p$ (for every message $a$).

## Example

$!(1, 2, a)\ !(1, 2, b)\ ?(2, 1, a)\ ?(2, 1, b)$ is 2-bounded but not 1-bounded.

# Bounded MSCs

## Definition (Universally bounded MSCs)

Let $B \in \mathbb{N}$ and $B > 0$. An MSC $M \in \mathbb{M}$ is called universally $B$-bounded ($\forall B$-bounded, for short) if

$$Lin(M) \ = \ Lin^B(M)$$

where $Lin^B(M) := \{w \in Lin(M) \mid w \text{ is } B\text{-bounded}\}$.

## Intuition

MSC $M$ is $\forall B$-bounded if all its linearizations are $B$-bounded.

So: if $M$ is $B$-bounded, then a buffer capacity $B$ is sufficient for all possible runs of MSC $M$.

# Bounded MSCs

## Definition (Existentially bounded MSCs)

Let $B \in \mathbb{N}$ and $B > 0$. An MSC $M \in \mathbb{M}$ is called existentially $B$-bounded ($\exists B$-bounded, for short) if $Lin(M) \cap Lin^B(M) \neq \varnothing$.

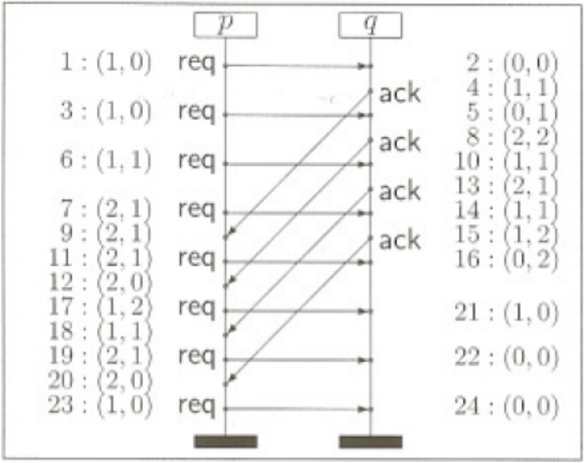
## Intuition

MSC $M$ is $\exists B$-bounded if at least one linearization is $B$-bounded.

## Consequence

The MSC $M$ can be "scheduled" in such a way that no channel ever contains more than $B$ messages.

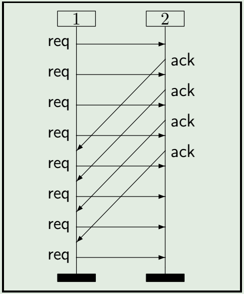
# Bounded MSCs

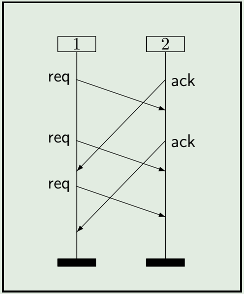


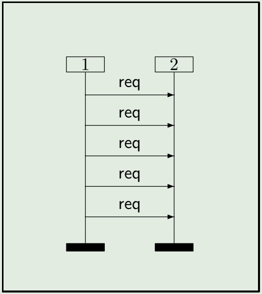

An $\exists 2$-bounded MSC with a corresponding justification

# Bounded MSCs

## Example




| | | |
|---|---|---|
| $\forall$4-bounded | $\forall$3-bounded | $\forall$5-bounded |
| $\exists$2-bounded | $\exists$1-bounded | $\exists$1-bounded |
| not $\exists$1-bounded | | |

# Bounded CFMs

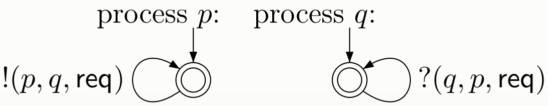
## Definition (Universally bounded CFM)

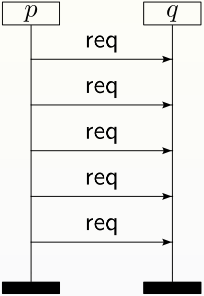
1. Let $B \in \mathbb{N}$ and $B > 0$. CFM $\mathcal{A}$ is *universally $B$-bounded* if each MSC in $\mathcal{L}(\mathcal{A})$ is $\forall B$-bounded.
2. CFM $\mathcal{A}$ is *universally bounded* if it is $\forall B$-bounded for some $B \in \mathbb{N}$ and $B > 0$.

## Definition (Existentially bounded CFM)

1. Let $B \in \mathbb{N}$ and $B > 0$. CFM $\mathcal{A}$ is *existentially $B$-bounded* if each MSC in $\mathcal{L}(\mathcal{A})$ is $\exists B$-bounded.
2. CFM $\mathcal{A}$ is *existentially bounded* if it is $\exists B$-bounded for some $B \in \mathbb{N}$ and $B > 0$.

# Example (1)

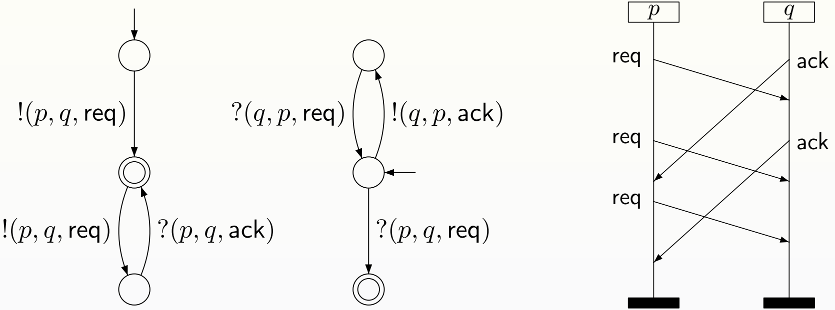



$\exists$1-bounded, but not $\forall B$-bounded for any $B$
so, not $\forall$-bounded.

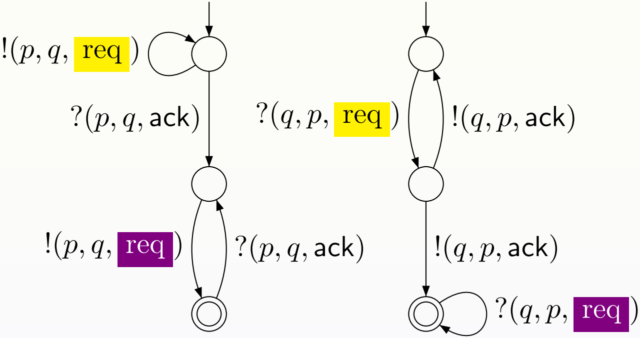
# Example (2)

process $p$:        process $q$:




$\exists$1-bounded, and $\forall$3-bounded

# Example (3)




$exists\lceil \frac{n}{2} \rceil$-bounded, but not $\forall B$-bounded for any $B$

# Justification

- Phase 1: process $p$ sends $n$ messages to $q$
  - messages of phase 1 are tagged with data req
- ... and waits for the first acknowledgement of $q$
- Phase 2: each ack is directly answered by $p$ by another message
  - messages of phase 2 are tagged with data req
- So, $p$ sends $2n$ reqs to $q$ and $q$ sends $n$ acks
  - existentially $\lceil \frac{n}{2} \rceil$-bounded, but not $\forall$-bounded
  - $q$ starts to send acks after $\lceil \frac{n}{2} \rceil$ request have been sent by $p$
  - after $n$ sends, process $p$ receives the first ack; then phase 2 starts
  - in phase 2, process $p$ and $q$ keep sending and receiving messages "in sync"
- Note: the CFM is also non-deterministic, and may deadlock. Why?

# Emptiness is decidable for ∃-bounded CFMs

**Theorem:** [Genest *et. al*, 2006]

For any ∃-bounded CFM, the emptiness problem is decidable (and is PSPACE-complete).

**Note:**

This decision problem is undecidable for arbitrary CFM, and is obviously decidable for ∀-bounded CFMs, as ∀-bounded CFMs have finitely many configurations, and thus one can check whether a configuration $(s, \eta_\varepsilon)$ with $s \in F$ is reachable by a simple graph analysis.

# Some (un)decidability results

## Undecidable

The following problems on CFM $\mathcal{A}$ are all undecidable:

1. Is CFM $\mathcal{A}$ universally bounded?
2. For $B \in \mathbb{N}$ and $B > 0$, is $\mathcal{A}$ $\forall B$-bounded?
3. Is CFM $\mathcal{A}$ existentially bounded?
4. For $B \in \mathbb{N}$ and $B > 0$, is $\mathcal{A}$ $\exists B$-bounded?

the proofs of all these facts are left as an exercise

# Deadlocks

## Deadlock-free CFMs

$(\overline{s}, \eta) \in Conf_{\mathcal{A}}$ is a deadlock configuration of CFM $\mathcal{A}$ if there is no 'accepting" configuration $(\overline{s}', \eta') \in F \times \{\eta_\varepsilon\}$ such that $(\overline{s}, \eta) \Longrightarrow^*_{\mathcal{A}} (\overline{s}', \eta')$.

CFM $\mathcal{A}$ is deadlock-free whenever it has no reachable deadlock configuration.

## Checking deadlock-freeness is undecidable

The decision problem: Is CFM $\mathcal{A}$ deadlock free? is undecidable.

## Checking $B$-boundedness for deadlock-free CFMs is decidable

The decision problem is decidable: for deadlock-free CFM $\mathcal{A}$ and $B \in \mathbb{N}$ with $B > 0$, is $\mathcal{A}$ $\forall B$-bounded?