Foundations of Informatics: a Bridging Course

Week 4: Formal Languages and Semantics
Part A: Regular Languages
b-it Bonn, 16-20 March 2015

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http://moves.rwth-aachen.de/teaching/ws-1415/foi/

Organisation

- Schedule:
  - lecture 9:00-10:30, 11:00-12:30 (Mon-Thu)
    - 10:00-11:30, 11:45-13:15?
  - exercises 14:00-14:45, 15:15-16:00 (Mon-Thu)
    - 14:00-15:30?
- Technical Writing exam on Tuesday morning
  - Tuesday afternoon session?
- Bridging Course exam on Friday, 20 March 2015, 13:00-16:00, b-it Rheinsaal
  - Friday morning session?
- Please ask questions!

Overview of Week 4

1. Regular Languages
2. Context-Free Languages

Literature

- J.E. Hopcroft, R. Motwani, J.D. Ullmann: *Introduction to Automata Theory, Languages, and Computation*, 2nd ed., Addison-Wesley, 2001
- A. Asteroth, C. Baier: *Theoretische Informatik*, Pearson Studium, 2002 [in German]
- `http://www.jflap.org/`
  (software for experimenting with formal languages and automata)

Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

Formal Languages

Words and Languages

- Computer systems transform data
- Data encoded as (binary) words

$\Longrightarrow$ Data sets = sets of words = formal languages,
data transformations = functions on words

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- Data encoded as (binary) words

$\Longrightarrow$ Data sets = sets of words = formal languages,
data transformations = functions on words

Example A.1

$$\mathit{Java} = \{\text{all valid Java programs}\},$$

$$\mathit{Compiler} : \mathit{Java} \to \mathit{Bytecode}$$

Formal Languages

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An alphabet is a finite, non-empty set of symbols (“letters”).

$\Sigma, \Gamma, \ldots$ denote alphabets

$a, b, \ldots$ denote letters

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1. Boolean alphabet $\mathbb{B} := \{0, 1\}$

Formal Languages

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2. Latin alphabet $\Sigma_{\mathrm{latin}} := \{a, b, c, \ldots, z\}$

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2. Latin alphabet $\Sigma_{\mathrm{latin}} := \{a, b, c, \ldots, z\}$
3. Keyboard alphabet Σ_{key}
4. Morse alphabet $\Sigma_{\mathrm{morse}} := \{\cdot, -, \sqcup\}$

Formal Languages

Words

Definition A.4

- A word is a finite sequence of letters from a given alphabet Σ.
- Σ^* is the set of all words over Σ.

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- The concatenation of two words $v = a_1 \dots a_m$ ($m \in \mathbb{N}$) and $w = b_1 \dots b_n$ ($n \in \mathbb{N}$) is the word
$$v \cdot w := a_1 \dots a_m b_1 \dots b_n$$
(often written as vw).
- Thus: $w \cdot \varepsilon = \varepsilon \cdot w = w$.

Formal Languages

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- A prefix/suffix v of a word w is an initial/trailing part of w, i.e., $w = vv'/w = v'v$ for some $v' \in \Sigma^*$.
- If $w = a_1 \dots a_n$, then $w^R := a_n \dots a_1$.

Formal Languages I

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A set of words $L \subseteq \Sigma^*$ is called a (formal) language over Σ.

Formal Languages

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Example A.6

1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101

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1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101
2. over $\Sigma = \{\mathsf{I}, \mathsf{V}, \mathsf{X}, \mathsf{L}, \mathsf{C}, \mathsf{D}, \mathsf{M}\}$: set of all valid roman numbers

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2. over $\Sigma = \{\mathrm{I}, \mathrm{V}, \mathrm{X}, \mathrm{L}, \mathrm{C}, \mathrm{D}, \mathrm{M}\}$: set of all valid roman numbers
3. over Σ_{key}: set of all valid Java programs

Formal Languages

Formal Languages II

Seen:

- Basic notions: alphabets, words
- Formal languages as sets of words

Formal Languages

Formal Languages II

Seen:

- Basic notions: alphabets, words
- Formal languages as sets of words

Open:

- Description of computations on words?

Finite Automata

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Regular Expressions

Minimisation of DFA

Outlook

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Example: Pattern Matching

Example A.7 (Pattern 1101)

1. Read Boolean string bit-by-bit
2. Test whether it contains 1101
3. Idea: remember which (initial) part of 1101 has been recognised
4. Five prefixes: ε, 1, 11, 110, 1101
5. Diagram: on the board

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What we used:

- finitely many (storage) states
- an initial state
- for every current state and every input symbol: a new state
- a successful state

Finite Automata

Deterministic Finite Automata I

Definition A.8

A deterministic finite automaton (DFA) is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$$

where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final (or: accepting) states

Finite Automata

Deterministic Finite Automata II

Example A.9

Pattern matching (Example A.7):

- $Q = \{q_0, \ldots, q_4\}$
- $\Sigma = \mathbb{B} = \{0, 1\}$
- $\delta : Q \times \Sigma \to Q$ on the board
- $F = \{q_4\}$

Finite Automata

Deterministic Finite Automata II

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- $\Sigma = \mathbb{B} = \{0, 1\}$
- $\delta : Q \times \Sigma \to Q$ on the board
- $F = \{q_4\}$

Graphical Representation of DFA:

- states $\Longrightarrow$ nodes
- $\delta(q, a) = q' \Longrightarrow q \xrightarrow{a} q'$
- initial state: incoming edge without source state
- final state(s): double circle

Finite Automata

Acceptance by DFA I

Definition A.10

Let $\langle Q, \Sigma, \delta, q_0, F\rangle$ be a DFA. The extension of $\delta : Q \times \Sigma \to Q$,

$$\delta^* : Q \times \Sigma^* \to Q,$$

is defined by

$$\delta^*(q, w) := \text{state after reading } w \text{ starting from } q.$$

Formally:

$$\delta^*(q, w) := \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), v) & \text{if } w = av \end{cases}$$

Thus: if $w = a_1 \dots a_n$ and $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$, then $\delta^*(q, w) = q_n$

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Thus: if $w = a_1 \dots a_n$ and $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$, then $\delta^*(q, w) = q_n$

Example A.11

Pattern matching (Example A.9): on the board

Finite Automata

Acceptance by DFA II

Definition A.12

- $\mathfrak{A}$ accepts $w \in \Sigma^*$ if $\delta^*(q_0, w) \in F$.
- The language recognised (or: accepted) by $\mathfrak{A}$ is

$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}.$$

- A language $L \subseteq \Sigma^*$ is called DFA-recognisable if there exists some DFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.
- Two DFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if

$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2).$$

Acceptance by DFA III

Example A.13

1. The set of all bit strings containing 1101 is recognised by the automaton from Example A.9.

Acceptance by DFA III

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2. Two (equivalent) automata recognising the language

$$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\}:$$

   on the board

Acceptance by DFA III

Example A.13

1. The set of all bit strings containing 1101 is recognised by the automaton from Example A.9.
2. Two (equivalent) automata recognising the language

   $$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\} :$$

   on the board
3. An automaton which recognises

   $$\{w \in \{0, \dots, 9\}^* \mid \text{value of } w \text{ divisible by } 3\}$$

   Idea: test whether sum of digits is divisible by 3 – one state for each residue class (on the board)

Finite Automata

Deterministic Finite Automata

Seen:

- Deterministic finite automata as a model of simple sequential computations
- Recognisability of formal languages by automata

Finite Automata

Deterministic Finite Automata

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- Recognisability of formal languages by automata

Open:

- Composition and transformation of automata?
- Which languages are recognisable, which are not (alternative characterisation)?
- Language definition $\mapsto$ automaton and vice versa?

Finite Automata

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Outlook

Operations on Languages

Simplest case: Boolean operations (complement, intersection, union)

Question

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two DFA with $L(\mathfrak{A}_1) = L_1$ and $L(\mathfrak{A}_2) = L_2$.
Can we construct automata which recognise

- $\overline{L_1}$ $(:= \Sigma^* \setminus L_1)$,
- $L_1 \cap L_2$, and
- $L_1 \cup L_2$?

Finite Automata

Language Complement

Theorem A.14

If $L \subseteq \Sigma^*$ *is DFA-recognisable, then so is* $\overline{L}$*.*

Language Complement

Theorem A.14

If $L \subseteq \Sigma^*$ is DFA-recognisable, then so is $\overline{L}$.

Proof.

Let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA such that $L(\mathfrak{A}) = L$. Then:

$$w \in \overline{L} \iff w \notin L \iff \delta^*(q_0, w) \notin F \iff \delta^*(q_0, w) \in Q \setminus F.$$

Thus, $\overline{L}$ is recognised by the DFA $\langle Q, \Sigma, \delta, q_0, Q \setminus F \rangle$. □

Finite Automata

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Example A.15

on the board

Language Intersection I

Theorem A.16

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cap L_2$.

Finite Automata

Language Intersection I

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Proof.

Let $\mathfrak{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathfrak{A}_i) = L_i$ $(i = 1, 2)$. The new automaton $\mathfrak{A}$ has to accept w iff $\mathfrak{A}_1$ and $\mathfrak{A}_2$ accept w

Idea: let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ run in parallel

- use pairs of states $(q_1, q_2) \in Q_1 \times Q_2$
- start with both components in initial state
- a transition updates both components independently
- for acceptance both components need to be in a final state

☐

Language Intersection II

Proof (continued).

Formally: let the product automaton

$$\mathfrak{A} := \langle Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), F_1 \times F_2 \rangle$$

be defined by

$$\delta((q_1, q_2), a) := (\delta_1(q_1, a), \delta_2(q_2, a)) \text{ for every } a \in \Sigma.$$

Language Intersection II

Proof (continued).

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This definition yields (for every $w \in \Sigma^*$):

$$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \qquad (*)$$

Finite Automata

Language Intersection II

Proof (continued).

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Thus we have:

$$\begin{aligned} & \mathfrak{A} \text{ accepts } w \\ \iff & \delta^*((q_0^1, q_0^2), w) \in F_1 \times F_2 \\ \overset{(*)}{\iff} & (\delta_1^*(q_0^1, w), \delta_2^*(q_0^2, w)) \in F_1 \times F_2 \\ \iff & \delta_1^*(q_0^1, w) \in F_1 \text{ and } \delta_2^*(q_0^2, w) \in F_2 \\ \iff & \mathfrak{A}_1 \text{ accepts } w \text{ and } \mathfrak{A}_2 \text{ accepts } w \end{aligned}$$

☐

Example A.17

on the board

Finite Automata

Language Union

Theorem A.18

If $L_1, L_2 \subseteq \Sigma^*$ *are DFA-recognisable, then so is* $L_1 \cup L_2$.

Finite Automata

Language Union

Theorem A.18

If $L_1, L_2 \subseteq \Sigma^*$ *are DFA-recognisable, then so is* $L_1 \cup L_2$*.*

Proof.

Let $\mathfrak{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathfrak{A}_i) = L_i$ $(i = 1, 2)$. The new automaton $\mathfrak{A}$ has to accept w iff $\mathfrak{A}_1$ or $\mathfrak{A}_2$ accepts w.

Finite Automata

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Idea: reuse product construction
Construct $\mathfrak{A}$ as before but choose as final states those pairs $(q_1, q_2) \in Q_1 \times Q_2$ with $q_1 \in F_1$ or $q_2 \in F_2$. Thus the set of final states is given by

$$F := (F_1 \times Q_2) \cup (Q_1 \times F_2).$$

□

Language Concatenation

Definition A.19

The concatenation of two languages $L_1, L_2 \subseteq \Sigma^*$ is given by

$$L_1 \cdot L_2 := \{v \cdot w \in \Sigma^* \mid v \in L_1, w \in L_2\}.$$

Abbreviations: $w \cdot L := \{w\} \cdot L$, $L \cdot w := L \cdot \{w\}$

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Example A.20

1. If $L_1 = \{101, 1\}$ and $L_2 = \{011, 1\}$, then
$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$

Finite Automata

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Example A.20

1. If $L_1 = \{101, 1\}$ and $L_2 = \{011, 1\}$, then
$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$
2. If $L_1 = 00 \cdot \mathbb{B}^*$ and $L_2 = 11 \cdot \mathbb{B}^*$, then
$$L_1 \cdot L_2 = \{w \in \mathbb{B}^* \mid w \text{ has prefix 00 and contains 11}\}.$$

Finite Automata

DFA-Recognisability of Concatenation

Conjecture

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cdot L_2$.

Finite Automata

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Proof (attempt).

Let $\mathfrak{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathfrak{A}_i) = L_i$ $(i = 1, 2)$. The new automaton $\mathfrak{A}$ has to accept w iff a prefix of w is recognised by $\mathfrak{A}_1$, and if $\mathfrak{A}_2$ accepts the remaining suffix.

Idea: choose $Q := Q_1 \cup Q_2$ where each $q \in F_1$ is identified with q_0^2

But: on the board □

Finite Automata

DFA-Recognisability of Concatenation

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But: on the board □

Conclusion

Required: automata model where the successor state (for a given state and input symbol) is not unique

Language Iteration

Definition A.21

- The nth power of a language $L \subseteq \Sigma^*$ is the n-fold concatenation of L with itself ($n \in \mathbb{N}$):
$$L^n := \underbrace{L \cdot \ldots \cdot L}_{n \text{ times}} = \{w_1 \ldots w_n \mid \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$
Inductively: $L^0 := \{\varepsilon\}$, $L^{n+1} := L^n \cdot L$
- The iteration (or: Kleene star) of L is
$$L^* := \bigcup_{n \in \mathbb{N}} L^n = \{w_1 \ldots w_n \mid n \in \mathbb{N}, \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$

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Remarks:

- we always have $\varepsilon \in L^*$ (since $L^0 \subseteq L^*$ and $L^0 = \{\varepsilon\}$)
- $w \in L^*$ iff $w = \varepsilon$ or if w can be decomposed into $n \geq 1$ subwords $v_1, \ldots, v_n$ (i.e., $w = v_1 \cdot \ldots \cdot v_n$) such that $v_i \in L$ for every $1 \leq i \leq n$
- again we would suspect that the iteration of a DFA-recognisable language is DFA-recognisable, but there is no simple (deterministic) construction

Operations on Languages and Automata

Seen:

- Operations on languages:
  - complement
  - intersection
  - union
  - concatenation
  - iteration
- DFA constructions for:
  - complement
  - intersection
  - union

Finite Automata

Operations on Languages and Automata

Seen:

- Operations on languages:
  - complement
  - intersection
  - union
  - concatenation
  - iteration
- DFA constructions for:
  - complement
  - intersection
  - union

Open:

- Automata model for (direct implementation of) concatenation and iteration?

Finite Automata

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

Nondeterministic Finite Automata I

Idea:

- for a given state and a given input symbol, several transitions (or none at all) are possible
- an input word generally induces several state sequences (“runs”)
- the word is accepted if at least one accepting run exists

Nondeterministic Finite Automata I

Idea:

- for a given state and a given input symbol, several transitions (or none at all) are possible
- an input word generally induces several state sequences (“runs”)
- the word is accepted if at least one accepting run exists

Advantages:

- simplifies representation of languages
  (example: $\mathbb{B}^* \cdot 1101 \cdot \mathbb{B}^*$; on the board)
- yields direct constructions for concatenation and iteration of languages
- more adequate modeling of systems with nondeterministic behaviour (communication protocols, multi-agent systems, ...)

Nondeterministic Finite Automata II

Definition A.22

A nondeterministic finite automaton (NFA) is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

Nondeterministic Finite Automata II

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- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

Remarks:

- $(q, a, q') \in \Delta$ usually written as $q \xrightarrow{a} q'$
- every DFA can be considered as an NFA ($(q, a, q') \in \Delta \iff \delta(q, a) = q'$)

Acceptance by NFA

Definition A.23

- Let $w = a_1 \dots a_n \in \Sigma^*$.
- A w-labelled $\mathfrak{A}$-run from q_1 to q_2 is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots p_{n-1} \xrightarrow{a_n} p_n$$

  such that $p_0 = q_1$, $p_n = q_2$, and $(p_{i-1}, a_i, p_i) \in \Delta$ for every $1 \leq i \leq n$ (we also write: $q_1 \xrightarrow{w} q_2$).
- $\mathfrak{A}$ accepts w if there is a w-labelled $\mathfrak{A}$-run from q_0 to some $q \in F$
- The language recognised by $\mathfrak{A}$ is

$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}.$$

- A language $L \subseteq \Sigma^*$ is called NFA-recognisable if there exists a NFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.
- Two NFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Finite Automata

Acceptance Test for NFA

Algorithm A.24 (Acceptance Test for NFA)

Input: *NFA* $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$, $w \in \Sigma^*$

Question: $w \in L(\mathfrak{A})$*?*

Procedure: *Computation of the reachability set*

$$R_{\mathfrak{A}}(w) := \{q \in Q \mid q_0 \xrightarrow{w} q\}$$

Iterative procedure for $w = a_1 \dots a_n$*:*

1. *let* $R_{\mathfrak{A}}(\varepsilon) := \{q_0\}$
2. *for* $i := 1, \dots, n$*: let*
   $R_{\mathfrak{A}}(a_1 \dots a_i) := \{q \in Q \mid \exists p \in R_{\mathfrak{A}}(a_1 \dots a_{i-1}) : p \xrightarrow{a_i} q\}$

Output: *"yes" if* $R_{\mathfrak{A}}(w) \cap F \neq \emptyset$*, otherwise "no"*

Remark: this algorithm solves the word problem for NFA

Finite Automata

Acceptance Test for NFA

Algorithm A.24 (Acceptance Test for NFA)

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```

Output: *"yes" if* $R_\mathfrak{A}(w) \cap F \neq \emptyset$, *otherwise "no"*

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Example A.25

on the board

NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

Solution: admit empty word ε as transition label

Finite Automata

ε-NFA

Definition A.26

A nondeterministic finite automaton with ε-transitions (ε-NFA) is of the form $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma_\varepsilon \times Q$ is the transition relation where $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

Remarks:

- every NFA is an ε-NFA
- definitions of runs and acceptance: in analogy to NFA

Finite Automata

ε-NFA

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Remarks:

- every NFA is an ε-NFA
- definitions of runs and acceptance: in analogy to NFA

Example A.27

on the board

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognisable, then so is $L_1 \cdot L_2$.

Finite Automata

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognisable, then so is $L_1 \cdot L_2$.

Proof (idea).

on the board ☐

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognisable, then so is $L_1 \cdot L_2$.

Proof (idea).

on the board ☐

Theorem A.29

If $L \subseteq \Sigma^*$ is ε-NFA-recognisable, then so is L^*.

Proof.

see Theorem A.47 ☐

Finite Automata

Syntax Diagrams as ε-NFA

Syntax diagrams (without recursive calls) can be interpreted as ε-NFA

Example A.30

decimal numbers (on the board)

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε-NFA (Definition A.26)

Finite Automata

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε-NFA (Definition A.26)

From the definitions we immediately obtain:

Corollary A.31

1. *Every DFA-recognisable language is NFA-recognisable.*
2. *Every NFA-recognisable language is ε-NFA-recognisable.*

Finite Automata

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε-NFA (Definition A.26)

From the definitions we immediately obtain:

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1. *Every DFA-recognisable language is NFA-recognisable.*
2. *Every NFA-recognisable language is ε-NFA-recognisable.*

Goal: establish reverse inclusions

From NFA to DFA I

Theorem A.32

Every NFA can be transformed into an equivalent DFA.

Finite Automata

From NFA to DFA I

Theorem A.32

Every NFA can be transformed into an equivalent DFA.

Proof.

Idea: let the DFA operate on sets of states (“powerset construction”)

- Initial state of DFA := $\{$initial state of NFA$\}$
- $P \xrightarrow{a} P'$ in DFA iff there exist $q \in P, q' \in P'$ such that $q \xrightarrow{a} q'$ in NFA
- P final state in DFA iff it contains some final state of NFA

□

From NFA to DFA II

Proof (continued).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F\rangle$ be a NFA.

Powerset construction of $\mathfrak{A}' = \langle Q', \Sigma, \delta', q_0', F'\rangle$:

- $Q' := 2^Q := \{P \mid P \subseteq Q\}$
- $\delta' : Q' \times \Sigma \to Q'$ with
$$q \in \delta'(P, a) \iff \text{there exists } p \in P \text{ such that } (p, a, q) \in \Delta$$
- $q_0' := \{q_0\}$
- $F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$

This yields

$$q_0 \xrightarrow{w} q \text{ in } \mathfrak{A} \iff q \in \delta'^*(\{q_0\}, w) \text{ in } \mathfrak{A}'$$

and thus

$$\mathfrak{A} \text{ accepts } w \iff \mathfrak{A}' \text{ accepts } w$$

☐

From NFA to DFA II

Proof (continued).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a NFA.
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and thus
$$\mathfrak{A} \text{ accepts } w \iff \mathfrak{A}' \text{ accepts } w$$
☐

Example A.33

on the board

Finite Automata

From ε-NFA to NFA

Theorem A.34

Every ε-NFA can be transformed into an equivalent NFA.

From ε-NFA to NFA

Theorem A.34

Every ε-NFA can be transformed into an equivalent NFA.

Proof (idea).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F\rangle$ be a ε-NFA. We construct the NFA $\mathfrak{A}'$ by eliminating all ε-transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}^* q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}^* r$ in $\mathfrak{A}$, then $p \xrightarrow{a} r$ in $\mathfrak{A}'$. Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}^* q \in F$ in $\mathfrak{A}$, and $F' := F$ otherwise. ☐

Finite Automata

From ε-NFA to NFA

Theorem A.34

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Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F\rangle$ be a ε-NFA. We construct the NFA $\mathfrak{A}'$ by eliminating all ε-transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}{}^{*} q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}{}^{*} r$ in $\mathfrak{A}$, then $p \xrightarrow{a} r$ in $\mathfrak{A}'$. Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}{}^{*} q \in F$ in $\mathfrak{A}$, and $F' := F$ otherwise. ☐

Example A.35

on the board

Finite Automata

From ε-NFA to NFA

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Example A.35

on the board

Corollary A.36

All types of finite automata recognise the same class of languages.

Finite Automata

Nondeterministic Finite Automata

Seen:

- Definition of ε-NFA
- Determinisation of (ε-)NFA

Finite Automata

Nondeterministic Finite Automata

Seen:

- Definition of ε-NFA
- Determinisation of (ε-)NFA

Open:

- More decidablity results

Finite Automata

Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

Finite Automata

The Word Problem Revisited

Definition A.37

The word problem for DFA is specified as follows:

Given a DFA $\mathfrak{A}$ and a word $w \in \Sigma^*$, decide whether

$$w \in L(\mathfrak{A}).$$

Finite Automata

The Word Problem Revisited

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$$w \in L(\mathfrak{A}).$$

As we have seen (Def. A.10, Alg. A.24, Thm. A.34):

Theorem A.38

The word problem for DFA (NFA, ε-NFA) is decidable.

Finite Automata

The Emptiness Problem

Definition A.39

The emptiness problem for DFA is specified as follows:

Given a DFA $\mathfrak{A}$, decide whether $L(\mathfrak{A}) = \emptyset$.

Finite Automata

The Emptiness Problem

Definition A.39

The emptiness problem for DFA is specified as follows:

Given a DFA $\mathfrak{A}$, decide whether $L(\mathfrak{A}) = \emptyset$.

Theorem A.40

The emptiness problem for DFA (NFA, ε-NFA) is decidable.

Proof.

It holds that $L(\mathfrak{A}) \neq \emptyset$ iff in $\mathfrak{A}$ some final state is reachable from the initial state (simple graph-theoretic problem). □

Finite Automata

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Proof.

It holds that $L(\mathfrak{A}) \neq \emptyset$ iff in $\mathfrak{A}$ some final state is reachable from the initial state (simple graph-theoretic problem). ☐

Remark: important result for formal verification
(unreachability of bad [= final] states)

Finite Automata

The Equivalence Problem

Definition A.41

The equivalence problem for DFA is specified as follows:

Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether

$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2).$$

The Equivalence Problem

Definition A.41

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Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether

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Theorem A.42

The equivalence problem for DFA (NFA, ε-NFA) is decidable.

Proof.

$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$$

Finite Automata

The Equivalence Problem

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The equivalence problem for DFA (NFA, ε-NFA) is decidable.

Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff \quad & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \end{aligned}$$

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Proof.

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\iff\ & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \\
\iff\ & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) \cup (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset
\end{aligned}$$

Finite Automata

The Equivalence Problem

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\iff\quad & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1)\\
\iff\quad & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) \cup (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset\\
\iff\quad & \underbrace{\underbrace{\underbrace{(L(\mathfrak{A}_1) \cap \underbrace{\overline{L(\mathfrak{A}_2)}}_{\text{DFA-recognisable (Thm. A.14)}})}_{\text{DFA-recognisable (Thm. A.16)}} \cup \underbrace{(L(\mathfrak{A}_2) \cap \underbrace{\overline{L(\mathfrak{A}_1)}}_{\text{DFA-recognisable (Thm. A.14)}})}_{\text{DFA-recognisable (Thm. A.16)}}}_{\text{DFA-recognisable (Thm. A.18)}} = \emptyset}_{\text{decidable (Thm. A.40)}} \qquad \square
\end{aligned}
$$

Finite Automata

Finite Automata

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Finite Automata

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Open:

- Non-algorithmic description of languages

Regular Expressions

Outline of Part A

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Regular Expressions

An Example

Example A.43

Consider the set of all words over $\Sigma := \{a, b\}$ which

1. start with one or three a symbols
2. continue with a (potentially empty) sequence of blocks, each containing at least one b and exactly two a's
3. conclude with a (potentially empty) sequence of b's

Corresponding regular expression:

$$(a + aaa)(\underbrace{bb^*ab^*ab^*}_{b\text{ before }a\text{'s}} + \underbrace{b^*abb^*ab^*}_{b\text{ between }a\text{'s}} + \underbrace{b^*ab^*abb^*}_{b\text{ after }a\text{'s}})^*b^*$$

Regular Expressions

Syntax of Regular Expressions

Definition A.44

The set of regular expressions over Σ is inductively defined by:

- $\emptyset$ and ε are regular expressions
- every $a \in \Sigma$ is a regular expression
- if α and β are regular expressions, then so are
  - $\alpha + \beta$
  - $\alpha \cdot \beta$
  - α^*

Regular Expressions

Syntax of Regular Expressions

Definition A.44

The set of regular expressions over Σ is inductively defined by:

- $\emptyset$ and ε are regular expressions
- every $a \in \Sigma$ is a regular expression
- if α and β are regular expressions, then so are
  - $\alpha + \beta$
  - $\alpha \cdot \beta$
  - α^*

Notation:

- $\cdot$ can be omitted
- * binds stronger than $\cdot$, $\cdot$ binds stronger than $+$
- α^+ abbreviates $\alpha \cdot \alpha^*$

Regular Expressions

Semantics of Regular Expressions

Definition A.45

Every regular expression α defines a language $L(\alpha)$:

$$\begin{aligned} L(\emptyset) &:= \emptyset \\ L(\varepsilon) &:= \{\varepsilon\} \\ L(a) &:= \{a\} \\ L(\alpha + \beta) &:= L(\alpha) \cup L(\beta) \\ L(\alpha \cdot \beta) &:= L(\alpha) \cdot L(\beta) \\ L(\alpha^*) &:= (L(\alpha))^* \end{aligned}$$

Regular Expressions

Semantics of Regular Expressions

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Every regular expression α defines a language $L(\alpha)$:

$$\begin{aligned} L(\emptyset) &:= \emptyset \\ L(\varepsilon) &:= \{\varepsilon\} \\ L(a) &:= \{a\} \\ L(\alpha + \beta) &:= L(\alpha) \cup L(\beta) \\ L(\alpha \cdot \beta) &:= L(\alpha) \cdot L(\beta) \\ L(\alpha^*) &:= (L(\alpha))^* \end{aligned}$$

A language L is called regular if it is definable by a regular expression, i.e., if $L = L(\alpha)$ for some regular expression α.

Regular Expressions

Regular Languages

Example A.46

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

Regular Expressions

Regular Languages

Example A.46

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

2. $\{a, b\}^*$ is regular since

$$L((a + b)^*) = (L(a + b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

Regular Expressions

Regular Languages

Example A.46

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

2. $\{a, b\}^*$ is regular since

$$L((a + b)^*) = (L(a + b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

3. The set of all words over $\{a, b\}$ containing *abb* is regular since

$$L((a + b)^* \cdot a \cdot b \cdot b \cdot (a + b)^*) = \{a, b\}^* \cdot \{abb\} \cdot \{a, b\}^*$$

Regular Expressions

Regular Languages and Finite Automata I

Theorem A.47 (Kleene's Theorem)

To each regular expression there corresponds an ε-NFA, and vice versa.

Regular Languages and Finite Automata I

Theorem A.47 (Kleene's Theorem)

To each regular expression there corresponds an ε-NFA, and vice versa.

Proof.

$\Longrightarrow$ using induction over the given regular expression α, we construct an ε-NFA $\mathfrak{A}_\alpha$

- with exactly one final state q_f
- without transitions into the initial state
- without transitions leaving the final state

(on the board)

$\Longleftarrow$ by solving a regular equation system (details omitted)

$\square$

Regular Languages and Finite Automata II

Corollary A.48

The following properties are equivalent:

- *L is regular*
- *L is DFA-recognisable*
- *L is NFA-recognisable*
- *L is ε-NFA-recognisable*

Implementation of Pattern Matching

Algorithm A.49 (Pattern Matching)

Input: *regular expression* α *and* $w \in \Sigma^*$

Question: *does* w *contain some* $v \in L(\alpha)$*?*

Procedure:
1. *let* $\beta := (a_1 + \ldots + a_n)^* \cdot \alpha$ *(for* $\Sigma = \{a_1, \ldots, a_n\}$*)*
2. *determine* ε*-NFA* $\mathfrak{A}_\beta$ *for* β
3. *eliminate* ε*-transitions*
4. *apply powerset construction to obtain DFA* $\mathfrak{A}$
5. *let* $\mathfrak{A}$ *run on* w

Output: *"yes" if* $\mathfrak{A}$ *passes through some final state, otherwise "no"*

Remark: in UNIX/LINUX implemented by `grep` and `lex`

Regular Expressions

Regular Expressions in UNIX (`grep`, `flex`, ...)

Syntax	Meaning
printable character	this character
`\n`, `\t`, `\123`, etc.	newline, tab, octal representation, etc.
`.`	any character except `\n`
`[`*Chars*`]`	one of *Chars*; ranges possible (“`0-9`”)
`[^`*Chars*`]`	none of *Chars*
`\\`, `\.`, `\[`, etc.	`\`, `.`, `[`, etc.
`"`*Text*`"`	*Text* without interpretation of `.`, `[`, `\`, etc.
`^`α	α at beginning of line
α`$`	α at end of line
α`?`	zero or one α
α`*`	zero or more α
α`+`	one or more α
α`{`*n*, *m*`}`	between *n* and *m* times α (“, *m*” optional)
`(`α`)`	α
$\alpha_1\alpha_2$	concatenation
$\alpha_1 \vert \alpha_2$	alternative

Regular Expressions

Regular Expressions

Seen:

- Definition of regular expressions
- Equivalence of regular and DFA-recognisable languages

Minimisation of DFA

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata
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Regular Expressions

Minimisation of DFA

Outlook

Minimisation of DFA

Motivation

Goal: space-efficient implementation of regular languages

Given: DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$

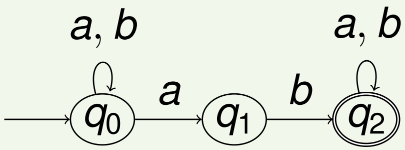
Wanted: DFA $\mathfrak{A}_{min} = \langle Q', \Sigma, \delta', q_0', F' \rangle$ such that $L(\mathfrak{A}_{min}) = L(\mathfrak{A})$ and $|Q'|$ minimal

Minimisation of DFA

State Equivalence

Example A.50

NFA for accepting $(a+b)^*ab(a+b)^*$:

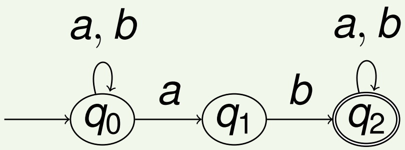


Minimisation of DFA

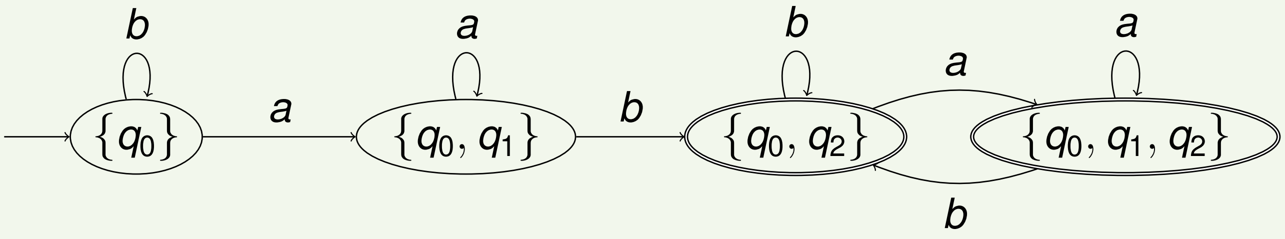
State Equivalence

Example A.50

NFA for accepting $(a+b)^*ab(a+b)^*$:



Powerset construction yields DFA $\mathfrak{A}$:



State Equivalence

Example A.50

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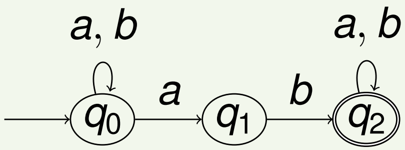

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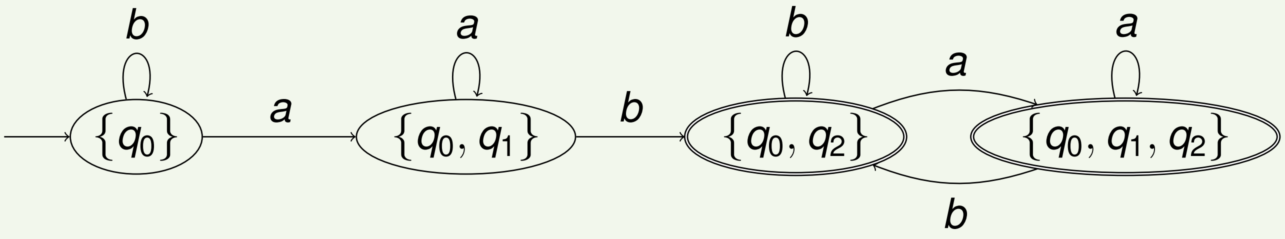


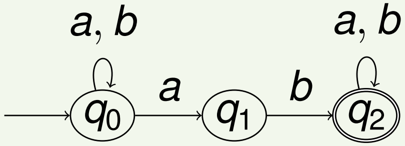
Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are equivalent

Minimisation of DFA

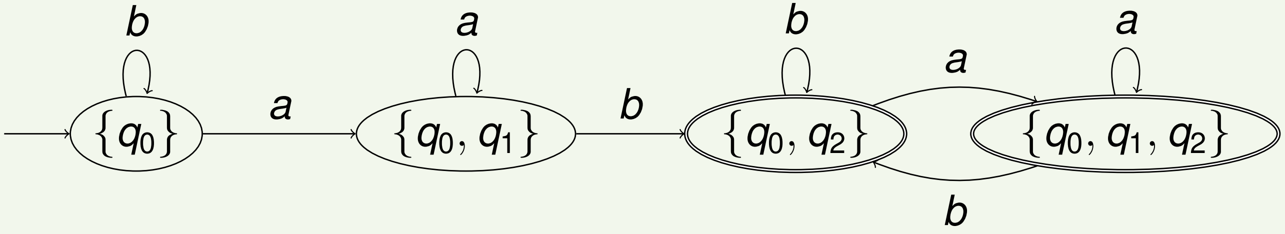
State Equivalence

Example A.50

NFA for accepting $(a+b)^*ab(a+b)^*$:



Powerset construction yields DFA $\mathfrak{A}$:



Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are equivalent

Definition A.51

Given DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F\rangle$, states $p, q \in Q$ are equivalent if

$$\forall w \in \Sigma^* : \delta^*(p, w) \in F \iff \delta^*(q, w) \in F.$$

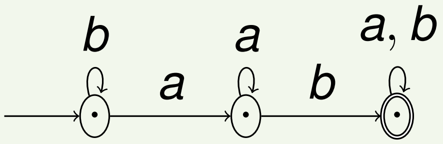
Minimisation of DFA

Minimisation

Minimisation: merging of equivalent states

Example A.52 (cf. Example A.50)

DFA after state merging:



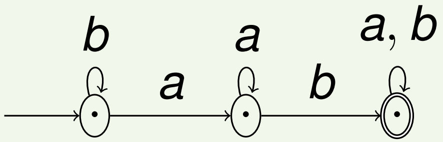
Minimisation of DFA

Minimisation

Minimisation: merging of equivalent states

Example A.52 (cf. Example A.50)

DFA after state merging:



Problem: identification of equivalent states

Approach: iterative computation of inequivalent states by refinement

Corollary A.53

$p, q \in Q$ are inequivalent if there exists $w \in \Sigma^*$ such that

$$\delta^*(p, w) \in F \text{ and } \delta^*(q, w) \notin F$$

(or vice versa, i.e., p and q can be distinguished by w)

Minimisation of DFA

Computing State (In-)Equivalence

Lemma A.54

Inductive characterisation of state inequivalence:

- $w = \varepsilon$: $p \in F, q \notin F \implies p, q$ *inequivalent (by* ε*)*
- $w = av$: p', q' *inequivalent (by* v*),* $p \xrightarrow{a} p', q \xrightarrow{a} q'$
  $\implies p, q$ *inequivalent (by* w*)*

Minimisation of DFA

Computing State (In-)Equivalence

Lemma A.54

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- $w = av$: p', q' *inequivalent (by* v*)*, $p \xrightarrow{a} p', q \xrightarrow{a} q'$
  $\implies p, q$ *inequivalent (by* w*)*

Algorithm A.55 (State Equivalence for DFA)

Input: *DFA* $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F\rangle$

Procedure: *Computation of "equivalence matrix" over* $Q \times Q$

1. *mark every pair* (p, q) *with* $p \in F, q \notin F$ *by* ε
2. *for every unmarked pair* (p, q) *and every* $a \in \Sigma$:
   if $(\delta(p, a), \delta(q, a))$ *marked by* v, *then mark* (p, q) *by* av
3. *repeat until no change*

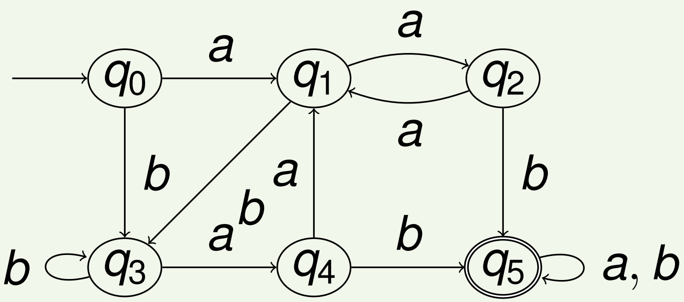
Output: *all equivalent (= unmarked) pairs of states*

Minimisation of DFA

Minimisation Example

Example A.56

Given DFA:



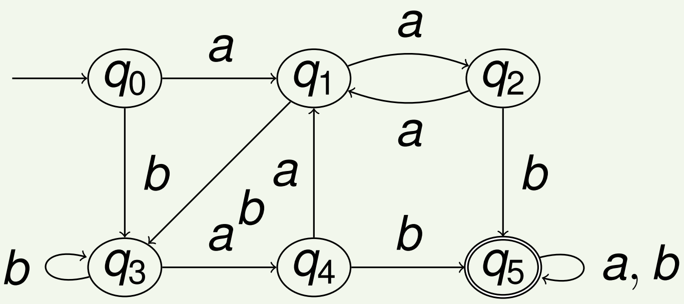
Equivalence matrix: on the board

Minimisation of DFA

Minimisation Example

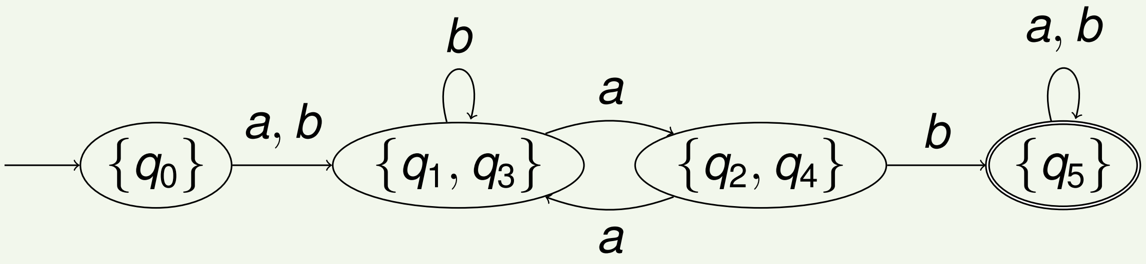
Example A.56

Given DFA:



Equivalence matrix: on the board

Resulting minimal DFA:



Minimisation of DFA

Correctness of Minimisation

Theorem A.57

For every DFA $\mathfrak{A}$,

$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Minimisation of DFA

Correctness of Minimisation

Theorem A.57

For every DFA $\mathfrak{A}$,

$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Remark: the minimal DFA is unique, in the following sense:

$$\forall \text{DFA } \mathfrak{A}, \mathfrak{B} : L(\mathfrak{A}) = L(\mathfrak{B}) \implies \mathfrak{A}_{min} \approx \mathfrak{B}_{min}$$

where $\approx$ refers to automata isomorphism (= identity up to naming of states)

Outlook

Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

Outlook

Outlook

- Pumping Lemma (to prove non-regularity of languages)
  - can be used to show that $\{a^n b^n \mid n \geq 1\}$ is not regular
- More language operations (homomorphisms, ...)
- Construction of scanners for compilers