Exercise Sheet 5: Denotational Semantics

Due date: May 20th. You can hand in your solutions at the start of the exercise class.

Exercise 1 (Denotational Semantics of a Recursive Function) 75%

Consider the following recursive program for $n \in \mathbb{Z}$:

$$\text{fac}(n) := \mathsf{if}\ (n = 0)\ \mathsf{then}\ \{1\}\ \mathsf{else}\ \{\text{fac}(n-1) * n\};$$

(a) [5%] Determine the functional $\Phi : (\mathbb{Z} \dashrightarrow \mathbb{Z}) \to (\mathbb{Z} \dashrightarrow \mathbb{Z})$ for $\text{fac}(n)$, as in the lecture.

(b) [20%] Show that Φ is monotonic and continuous.

(c) [10%] Show that the partial order $(\mathbb{Z} \dashrightarrow \mathbb{Z}, \sqsubseteq)$ is chain complete.

(d) [20%] Let $\mathfrak{C}[\![\text{fac}(n)]\!]$ be defined by $\text{fix}(\Phi)$. Compute the denotational semantics of $\text{fac}(3)$.

(e) [20%] Prove that the program fac calculates the factorial, i.e. $\text{fix}(\Phi)(n) = n!$ for any $n \in \mathbb{Z}$.

Exercise 2 (Denotational Semantics of Guarded Commands) 25%

Dijkstra's *guarded commands* are essentially of the form

$$\mathsf{do}\{b_1 \to c_1\ \ b_2 \to c_2\}$$

(where $b_1, b_2 \in \mathsf{BExp}$ and $c_1, c_2 \in \mathsf{Cmd}$). They form a natural generalisation of the `WHILE` loop:

> While at least one of the tests is true, the corresponding statement is executed. Here the satisfaction of both tests results in a non–deterministic choice of the command. The computation terminates as soon as neither of the tests is true.

(a) [10%] Which function on the natural numbers is computed by the following statement? Transform it to a *WHILE* statement.

$$\begin{array}{l} \mathsf{do}\{ \\ \quad x > y \to x := x - y \\ \quad y > x \to y := y - x \\ \quad \} \end{array}$$

(b) [15%] Let $b_1, b_2 \in \mathsf{BExp}$ be two mutually excluding tests (i.e., in no state both b_1 and b_2 are true) and $c_1, c_2 \in \mathsf{Cmd}$. How can the semantics of

$$\begin{array}{l} \mathsf{do}\{ \\ \quad b_1 \to c_1 \\ \quad b_2 \to c_2 \\ \quad \} \end{array}$$

be defined as the least fixpoint of a mapping

$$\Phi : (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma)?$$