Modeling and Verification of Probabilistic Systems

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Overview

Continuous Stochastic Logic

- CSL is a language for formally specifying properties over CTMCs.
- It is a branching-time temporal logic based on CTL.
- Formula interpretation is Boolean, i.e., a state satisfies a formula or not.
- Like in PCTL, the main operator is $\mathbb{P}_J(\varphi)$
  - where φ constrains the set of paths and J is a threshold on the probability.
  - it is the probabilistic counterpart of $\exists$ and $\forall$ path-quantifiers in CTL.
- The new features are a timed version of the next and until-operator.
  - $\bigcirc^I \Phi$ asserts that a transition to a Φ-state can be made at time $t \in I$.
  - $\Phi \, \mathsf{U}^I \, \Psi$ asserts that a Ψ-state can be reached via Φ-states at time $t \in I$.

CTMCs — A transition system perspective

Continuous-time Markov chain

A CTMC $\mathcal{C}$ is a tuple $(S, \mathbf{P}, r, \iota_{\text{init}}, AP, L)$ with:

- S is a countable nonempty set of states
- $\mathbf{P} : S \times S \rightarrow [0,1]$, transition probability function s.t. $\sum_{s'} \mathbf{P}(s,s') = 1$
- $r : S \rightarrow \mathbb{R}_{>0}$, rate assigning function
- $\iota_{\text{init}} : S \rightarrow [0,1]$, the initial distribution with $\sum\limits_{s \in S} \iota_{\text{init}}(s) = 1$
- AP is a set of atomic propositions.
- $L : S \rightarrow 2^{AP}$, the labeling function, assigning to state s, the set $L(s)$ of atomic propositions that are valid in s.

Residence time

The average residence time in state s is $\frac{1}{r(s)}$.

CSL syntax

[Baier, Katoen & Hermanns, 1999]

Continuous Stochastic Logic: Syntax

CSL consists of state- and path-formulas.

- CSL *state formulas* over the set AP obey the grammar:

$$\Phi \ ::= \ \text{true} \ \Big| \ a \ \Big| \ \Phi_1 \wedge \Phi_2 \ \Big| \ \neg\Phi \ \Big| \ \mathbb{P}_J(\varphi)$$

where $a \in AP$, φ is a path formula and $J \subseteq [0,1]$, $J \neq \varnothing$ is a non-empty interval.

- CSL *path formulae* are formed according to the following grammar:

$$\varphi \ ::= \ \bigcirc^I \Phi \ \Big| \ \Phi_1 \, \mathsf{U}^I \, \Phi_2$$

where Φ, Φ_1, and Φ_2 are state formulae and $I \subseteq \mathbb{R}_{\geqslant 0}$ an interval.

Abbreviate $\mathbb{P}_{[0,0.5]}(\varphi)$ by $\mathbb{P}_{\leqslant 0.5}(\varphi)$ and $\mathbb{P}_{]0,1]}(\varphi)$ by $\mathbb{P}_{>0}(\varphi)$.

Continuous Stochastic Logic

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where Φ, Φ_1, and Φ_2 are state formulae and $I \subseteq \mathbb{R}_{\geqslant 0}$ an interval.

Intuitive semantics

- $s_0 t_0 s_1 t_1 \ldots \models \Phi \, \mathsf{U}^I \, \Psi$ if Ψ is reached at $t \in I$ and prior to t, Φ holds.
- $s \models \mathbb{P}_J(\varphi)$ if probability that paths starting in s fulfill φ lies in J.

Overview

Derived operators

$$\Diamond\Phi \ = \ \text{true} \, \mathsf{U} \, \Phi$$

$$\Diamond^I\Phi \ = \ \text{true} \, \mathsf{U}^I \Phi$$

$$\mathbb{P}_{\leqslant p}(\Box\Phi) \ = \ \mathbb{P}_{>1-p}(\Diamond\neg\Phi)$$

$$\mathbb{P}_{(p,q)}(\Box^I\Phi) \ = \ \mathbb{P}_{[1-q,1-p]}(\Diamond^I\neg\Phi)$$

Paths in a CTMC

Timed paths

Paths in CTMC $\mathcal{C}$ are maximal (i.e., infinite) paths of alternating states and time instants:

$$\pi \;=\; s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} s_2 \cdots$$

such that $s_i \in S$ and $t_i \in \mathbb{R}_{>0}$. Let $Paths(\mathcal{C})$ be the set of paths in $\mathcal{C}$ and $Paths^*(\mathcal{C})$ the set of finite prefixes thereof.

Notations

- Let $\pi[i] := s_i$ denote the $(i{+}1)$-st state along the timed path π.
- Let $\pi\langle i\rangle := t_i$ the time spent in state s_i.
- Let $\pi@t$ be the state occupied in π at time $t \in \mathbb{R}_{\geqslant 0}$, i.e. $\pi@t := \pi[i]$ where i is the smallest index such that $\sum_{j=0}^{i} \pi\langle j\rangle \;>\; t$.

Example properties

- Transient probabilities to be in *goal* state at time point 4:

$$\mathbb{P}_{\geqslant 0.92}\left(\Diamond^{=4}\; goal\right)$$

- With probability $\geqslant 0.92$, a goal state is reached legally:

$$\mathbb{P}_{\geqslant 0.92}\left(\neg\, illegal \;\; \mathsf{U} \;\; goal\right)$$

- . . . in maximally 137 time units: $\mathbb{P}_{\geqslant 0.92}\left(\neg\, illegal \;\; \mathsf{U}^{\leqslant 137} \;\; goal\right)$
- . . . once there, remain there almost surely for the next 31 time units:

$$\mathbb{P}_{\geqslant 0.92}\left(\neg\, illegal \;\; \mathsf{U}^{\leqslant 137} \; \mathbb{P}_{=1}(\Box^{[0,31]} \; goal)\right)$$

CSL semantics (1)

Notation

$\mathcal{C}, s \models \Phi$ if and only if state-formula Φ holds in state s of CTMC $\mathcal{C}$.

Satisfaction relation for state formulas

The satisfaction relation $\models$ is defined for CSL state formulas by:

$$\begin{array}{lll}
s \models a & \text{iff} & a \in L(s) \\
s \models \neg\,\Phi & \text{iff} & \text{not } (s \models \Phi) \\
s \models \Phi \wedge \Psi & \text{iff} & (s \models \Phi) \text{ and } (s \models \Psi) \\
s \models \mathbb{P}_J(\varphi) & \text{iff} & Pr(s \models \varphi) \in J
\end{array}$$

where $Pr(s \models \varphi) \;=\; Pr_s\{\, \pi \in Paths(s) \mid \pi \models \varphi \,\}$.

This is as for PCTL, except that Pr is the probability measures on cylinder sets of timed paths in CTMC $\mathcal{C}$.

CSL semantics (2)

Satisfaction relation for path formulas

Let $\pi = s_0\; t_0\; s_1\; t_1\; s_2 \ldots$ be an infinite path in CTMC $\mathcal{C}$.

The satisfaction relation $\models$ is defined for state formulas by:

$$\begin{array}{lll}
\pi \models \bigcirc^{I}\, \Phi & \text{iff} & s_1 \models \Phi \;\wedge\; t_0 \in I \\
\pi \models \Phi\, \mathsf{U}^{I}\, \Psi & \text{iff} & \exists t \in I.\; ((\forall t' \in [0,t).\, \pi@t' \models \Phi) \;\wedge\; \pi@t \models \Psi)
\end{array}$$

Standard next- and until-operators

- $X\Phi \;\equiv\; \bigcirc^{I}\, \Phi$ with $I = \mathbb{R}_{\geqslant 0}$.
- $\Phi\, \mathsf{U}\, \Psi \;\equiv\; \Phi\, \mathsf{U}^{I}\, \Psi$ with $I = \mathbb{R}_{\geqslant 0}$.

Measurability

CSL measurability

For any CSL path formula φ and state s of CTMC $\mathcal{C}$, the set $\{ \pi \in Paths(s) \mid \pi \models \varphi \}$ is measurable.

Proof:

Rather straightforward; left as an exercise.

Overview

CSL model checking

CSL model checking problem

Input: a finite CTMC $\mathcal{C} = (S, \mathbf{P}, r, \iota_{\text{init}}, AP, L)$, state $s \in S$, and CSL state formula Φ

Output: yes, if $s \models \Phi$; no, otherwise.

Basic algorithm

In order to check whether $s \models \Phi$ do:

1. Compute the satisfaction set $Sat(\Phi) = \{ s \in S \mid s \models \Phi \}$.
2. This is done recursively by a bottom-up traversal of Φ's parse tree.
   - The nodes of the parse tree represent the subformulae of Φ.
   - For each node, i.e., for each subformula Ψ of Φ, determine $Sat(\Psi)$.
   - Determine $Sat(\Psi)$ as function of the satisfaction sets of its children:

     e.g., $Sat(\Psi_1 \wedge \Psi_2) = Sat(\Psi_1) \cap Sat(\Psi_2)$ and $Sat(\neg\Psi) = S \setminus Sat(\Psi)$.
3. Check whether state s belongs to $Sat(\Phi)$.

Core model checking algorithm

Propositional formulas

$Sat(\cdot)$ is defined by structural induction as follows:

$$\begin{aligned} Sat(\text{true}) &= S \\ Sat(a) &= \{ s \in S \mid a \in L(s) \}, \text{ for any } a \in AP \\ Sat(\Phi \wedge \Psi) &= Sat(\Phi) \cap Sat(\Psi) \\ Sat(\neg\Phi) &= S \setminus Sat(\Phi). \end{aligned}$$

Probabilistic operator $\mathbb{P}$

In order to determine whether $s \in Sat(\mathbb{P}_J(\varphi))$, the probability $Pr(s \models \varphi)$ for the event specified by φ needs to be established. Then

$$Sat(\mathbb{P}_J(\varphi)) = \{ s \in S \mid Pr(s \models \varphi) \in J \}.$$

Let us consider the computation of $Pr(s \models \varphi)$ for all possible φ.

The next-step operator

Recall that: $s \models \mathbb{P}_J(\bigcirc^I \Phi)$ if and only if $Pr(s \models \bigcirc^I \Phi) \in J$.

Lemma

$$Pr(s \models \bigcirc^I \Phi) \;=\; \underbrace{\left(e^{-r(s)\cdot \inf I} - e^{-r(s)\cdot \sup I}\right)}_{\text{probability to leave } s \text{ in interval } I} \cdot \sum_{s' \in Sat(\Phi)} \mathbf{P}(s, s').$$

Algorithm

Considering the above equation for all states simultaneously yields:

$$\big(Pr(s \models \bigcirc \Phi)\big)_{s \in S} \;=\; \mathbf{b}_I^T \cdot \mathbf{P}$$

with $\mathbf{b}_I$ is defined by $b_I(s) = e^{-r(s)\cdot \inf I} - e^{-r(s)\cdot \sup I}$ if $s \in Sat(\Phi)$ and 0 otherwise, and $\mathbf{b}_I^T$ is the transposed variant of $\mathbf{b}_I$.

Time-bounded until (1)

Recall that: $s \models \mathbb{P}_J(\Phi \, \mathrm{U}^{\leqslant t} \, \Psi)$ if and only if $Pr(s \models \Phi \, \mathrm{U}^{\leqslant t} \, \Psi) \in J$.

Lemma

Let $S_{=1} = Sat(\Psi)$, $S_{=0} = S \setminus (Sat(\Phi) \cup Sat(\Psi))$, and $S_? = S \setminus (S_{=0} \cup S_{=1})$. Then:

$$Pr(s \models \Phi \, \mathrm{U}^{\leqslant t} \, \Psi) \;=\; \begin{cases} 1 & \text{if } s \in S_{=1} \\ 0 & \text{if } s \in S_{=0} \\ \displaystyle\int_0^t \sum_{s' \in S} \mathbf{R}(s, s') \cdot e^{-r(s)\cdot x} \cdot Pr(s' \models \Phi \, \mathrm{U}^{\leqslant t-x} \, \Psi) \, dx & \text{otherwise} \end{cases}$$

This is a slight generalisation of the Volterra integral equation system for timed reachability.

Time-bounded until (2)

Let $S_{=1} = Sat(\Psi)$, $S_{=0} = S \setminus (Sat(\Phi) \cup Sat(\Psi))$, and $S_? = S \setminus (S_{=0} \cup S_{=1})$. Then:

$$Pr(s \models \Phi \, \mathrm{U}^{\leqslant t} \, \Psi) \;=\; \begin{cases} 1 & \text{if } s \in S_{=1} \\ 0 & \text{if } s \in S_{=0} \\ \displaystyle\int_0^t \sum_{s' \in S} \mathbf{R}(s, s') \cdot e^{-r(s)\cdot x} \cdot Pr(s' \models \Phi \, \mathrm{U}^{\leqslant t-x} \, \Psi) \, dx & \text{otherwise} \end{cases}$$

Recall lemma from the previous lecture

$$\underbrace{Pr(s \models \overline{F} \, \mathrm{U}^{\leqslant t} \, G)}_{\text{timed reachability in } \mathcal{C}} \;=\; \underbrace{Pr(s \models \Diamond^{=t} G)}_{\text{in } \mathcal{C}[F \cup G]} \;=\; \underbrace{\underline{p}(t) \text{ with } \underline{p}(0) = \mathbf{1}_s}_{\text{transient prob. in } \mathcal{C}[F \cup G]}.$$

Phrased using CSL state formulas

$$\underbrace{Pr(s \models \Phi \, \mathrm{U}^{\leqslant t} \, \Psi)}_{\text{timed reachability in } \mathcal{C}} \;=\; \underbrace{Pr(s \models \Diamond^{=t} \Psi)}_{\text{in } \mathcal{C}[Sat(\neg\Phi) \cup Sat(\Psi)]} \;=\; \underbrace{\underline{p}(t) \text{ with } \underline{p}(0) = \mathbf{1}_s}_{\mathcal{C}[Sat(\neg\Phi) \cup Sat(\Psi)]}.$$

Time-bounded until (3)

Algorithm for checking $Pr(s \models \Phi \, \mathrm{U}^{\leqslant t} \, \Psi) \in J$

1. If $t = \infty$, then use approach for until (as in PCTL): solve a system of linear equations.
2. Determine recursively $Sat(\Phi)$ and $Sat(\Psi)$.
3. Make all states in $S \setminus Sat(\Phi)$ and $Sat(\Psi)$ absorbing.
4. Uniformize the resulting CTMC with respect to its maximal rate.
5. Determine the transient probability at time t using s as initial distribution.
6. Return yes if transient probability of all Ψ-states lies in J, and no otherwise.

Time-bounded until (4)

Possible optimizations

1. Make all states in $S \setminus Sat(\exists(\Phi \,\mathrm{U}\, \Psi))$ absorbing.
2. Make all states in $Sat(\forall(\Phi \,\mathrm{U}\, \Psi))$ absorbing.
3. Replace the labels of all states in $S \setminus Sat(\exists(\Phi\Psi))$ by unique label zero.
4. Replace the labels of all states in $Sat(\forall(\Phi \,\mathrm{U}\, \Psi))$ by unique label one.
5. Perform bisimulation minimization on all states.

The last step collapses all states in $S \setminus Sat(\exists(\Phi \,\mathrm{U}\, \Psi))$ into a single state, and does the same with all states in $Sat(\forall(\Phi \,\mathrm{U}\, \Psi))$.

Preservation of CSL-formulas

Bisimulation and CSL-equivalence coincide

Let $\mathcal{C}$ be a finitely branching CTMC and s, t states in $\mathcal{C}$. Then:

$$s \sim_m t \quad \text{if and only if} \quad s \text{ and } t \text{ are CSL-equivalent.}$$

Remarks

If for CSL-formula Φ we have $s \models \Phi$ but $t \not\models \Phi$, then it follows $s \not\sim_m t$. A single CSL-formula suffices!

Preservation of CSL-formulas

Weak bisimulation and CSL-without-next-equivalence coincide

Let $\mathcal{C}$ be a finitely branching CTMC and s, t states in $\mathcal{C}$. Then:

$$s \approx_m t \quad \text{if and only if} \quad s \text{ and } t \text{ are CSL-without-next-equivalent.}$$

Here. CSL-without-next is the fragment of CSL where the next-operator $\bigcirc$ does not occur.

Remarks

If for CSL-without-next-formula Φ we have $s \models \Phi$ but $t \not\models \Phi$, then it follows $s \not\approx_m t$.

Uniformization and CSL

Uniformization and CSL

For any finite CTMC $\mathcal{C}$ with state space S, $r \geqslant \max\{\, r(s) \mid s \in S \,\}$ and Φ a CSL-without-next-formula:

$$Sat^{\mathcal{C}}(\Phi) \;=\; Sat^{\mathcal{C}'}(\Phi) \quad \text{where } \mathcal{C}' = unif(r, \mathcal{C}).$$

Uniformization and CSL

For any uniformized CTMC: CSL-equivalence coincides with CSL-without-next-equivalence.

Overview

Time complexity

Let $|\Phi|$ be the size of Φ, i.e., the number of logical and temporal operators in Φ.

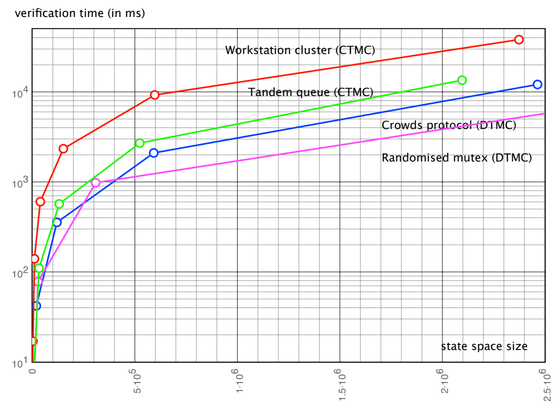
Time complexity of CSL model checking

For finite CTMC $\mathcal{C}$ and CSL state-formula Φ, the CSL model-checking problem can be solved in time

$$\mathcal{O}(\, poly(size(\mathcal{C})) \cdot t_{max} \cdot |\Phi|\,)$$

where $t_{max} \; = \; \max\{\, t \mid \Psi_1 \, \mathsf{U}^{\leqslant t} \Psi_2 \text{ occurs in } \Phi \,\}$ with and $t_{max} = 1$ if Φ does not contain a time-bounded until-operator.

Some practical verification times



- command-line tool MRMC ran on a Pentium 4, 2.66 GHz, 1 GB RAM laptop.
- CSL formulas are time-bounded until-formulas.

Overview

Summary

- CSL is a variant of PCTL with timed next and timed until.
- Sets of paths fulfilling CSL path-formula φ are measurable.
- CSL model checking is performed by a recursive descent over Φ.
- The timed next operator amounts to a single vector-matrix multiplication.
- The time-bounded until-operator $\mathsf{U}^{\leqslant t}$ is solved by uniformization.
- The worst-case time complexity is polynomial in the size of the CTMC and linear in the size of the formula.