Compiler Construction

Lecture 3: Lexical Analysis II (Extended Matching Problem)

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Summer Semester 2014

Outline

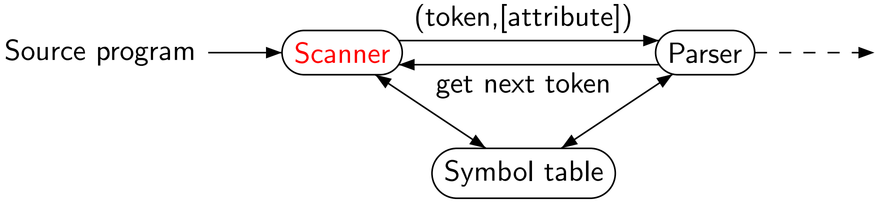
1. Recap: Lexical Analysis
2. Complexity Analysis of Simple Matching
3. The Extended Matching Problem
4. First-Longest-Match Analysis
5. Implementation of FLM Analysis

Lexical Analysis

Definition

The goal of lexical analysis is to decompose a source program into a sequence of lexemes and their transformation into a sequence of symbols.

The corresponding program is called a scanner (or lexer):



Example:

`...␣x1␣:=y2+␣1␣;␣...`

$$\Downarrow$$

$\ldots(\mathrm{id}, p_1)(\mathrm{gets},)(\mathrm{id}, p_2)(\mathrm{plus},)(\mathrm{int}, 1)(\mathrm{sem},)\ldots$

The DFA Method I

Known from *Formal Systems, Automata and Processes*:

Algorithm (DFA method)

Input: *regular expression* $\alpha \in RE_\Omega$, *input string* $w \in \Omega^*$

Procedure:
1. *using Kleene's Theorem, construct* $\mathfrak{A}_\alpha \in NFA_\Omega$ *such that* $L(\mathfrak{A}_\alpha) = [\![\alpha]\!]$
2. *apply powerset construction (cf. Definition 2.12) to obtain* $\mathfrak{A}'_\alpha = \langle Q', \Omega, \delta', q'_0, F' \rangle \in DFA_\Omega$ *with* $L(\mathfrak{A}'_\alpha) = L(\mathfrak{A}_\alpha) = [\![\alpha]\!]$
3. *solve the matching problem by deciding whether* $\delta'^*(q'_0, w) \in F'$

Output: *"yes" or "no"*

The DFA Method II

The powerset construction involves the following concept:

Definition (ε-closure)

Let $\mathfrak{A} = \langle Q, \Omega, \delta, q_0, F \rangle \in NFA_\Omega$. The ε-closure $\varepsilon(T) \subseteq Q$ of a subset $T \subseteq Q$ is defined by

- $T \subseteq \varepsilon(T)$ and
- if $q \in \varepsilon(T)$, then $\delta(q, \varepsilon) \subseteq \varepsilon(T)$

Definition (Powerset construction)

Let $\mathfrak{A} = \langle Q, \Omega, \delta, q_0, F \rangle \in NFA_\Omega$. The powerset automaton $\mathfrak{A}' = \langle Q', \Omega, \delta', q_0', F' \rangle \in DFA_\Omega$ is defined by

- $Q' := 2^Q$
- $\forall T \subseteq Q, a \in \Omega : \delta'(T, a) := \varepsilon\left(\bigcup_{q \in T} \delta(q, a)\right)$
- $q_0' := \varepsilon(\{q_0\})$
- $F' := \{T \subseteq Q \mid T \cap F \neq \emptyset\}$

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Complexity of DFA Method

1. in construction phase:
   - Kleene method: time and space $\mathcal{O}(|\alpha|)$
     (where $|\alpha| :=$ length of α)
   - Powerset construction: time and space $\mathcal{O}(2^{|\mathfrak{A}_\alpha|}) = \mathcal{O}(2^{|\alpha|})$
     (where $|\mathfrak{A}_\alpha| := \#$ of states of $\mathfrak{A}_\alpha$)

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2. at runtime:
   - Word problem: time $\mathcal{O}(|w|)$ (where $|w| :=$ length of w), space $\mathcal{O}(1)$ (but $\mathcal{O}(2^{|\alpha|})$ for storing DFA)

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$\Longrightarrow$ nice runtime behavior but memory requirements very high (and exponential time in construction phase)

The NFA Method

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input string $w \in \Omega^*$

Variables: $T \subseteq Q$, $a \in \Omega$

Procedure: $T := \varepsilon(\{q_0\})$;
while $w \neq \varepsilon$ **do**
  $a :=$ **head**(w);
  $T := \varepsilon\left(\bigcup_{q \in T} \delta(q, a)\right)$;
  $w :=$ **tail**(w)
od

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od

Output: *if* $T \cap F \neq \emptyset$ *then "yes" else "no"*

Complexity Analysis

For NFA method at runtime:

- Space: $\mathcal{O}(|\alpha|)$ (for storing NFA and T)
- Time: $\mathcal{O}(|\alpha| \cdot |w|)$
  (in the loop's body, $|T|$ states need to be considered)

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Comparison:

Method	Space	Time (for "$w \in [\![\alpha]\!]$?")
DFA	$\mathcal{O}(2^{|\alpha|})$	$\mathcal{O}(|w|)$
NFA	$\mathcal{O}(|\alpha|)$	$\mathcal{O}(|\alpha| \cdot |w|)$

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In practice:

- Exponential blowup of DFA method usually does not occur in "real" applications ($\Longrightarrow$ used in `[f]lex`)
- Improvement of NFA method: caching of transitions $\delta'(T, a)$
  $\Longrightarrow$ combination of both methods

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The Extended Matching Problem I

Definition 3.2

Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in RE_\Omega$ with $\varepsilon \notin [\![\alpha_i]\!] \neq \emptyset$ for every $i \in [n]$ $(= \{1, \ldots, n\})$. Let $\Sigma := \{T_1, \ldots, T_n\}$ be an alphabet of corresponding tokens and $w \in \Omega^+$. If $w_1, \ldots, w_k \in \Omega^+$ such that

- $w = w_1 \ldots w_k$ and
- for every $j \in [k]$ there exists $i_j \in [n]$ such that $w_j \in [\![\alpha_{i_j}]\!]$,

then

- $(w_1, \ldots, w_k)$ is called a decomposition and
- $(T_{i_1}, \ldots, T_{i_k})$ is called an analysis

of w w.r.t. $\alpha_1, \ldots, \alpha_n$.

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of w w.r.t. $\alpha_1, \ldots, \alpha_n$.

Problem 3.3 (Extended matching problem)

Given $\alpha_1, \ldots, \alpha_n \in RE_\Omega$ and $w \in \Omega^+$, decide whether there exists a decomposition of w w.r.t. $\alpha_1, \ldots, \alpha_n$ and determine a corresponding analysis.

The Extended Matching Problem II

Observation: neither the decomposition nor the analysis are uniquely determined

Example 3.4

1. $\alpha = a^+$, $w = aa$
   $\Longrightarrow$ two decompositions (aa) and (a, a) with respective (unique) analyses (T_1) and (T_1, T_1)

The Extended Matching Problem II

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Example 3.4

1. $\alpha = a^+$, $w = aa$
   $\Longrightarrow$ two decompositions (aa) and (a, a) with respective (unique) analyses (T_1) and (T_1, T_1)
2. $\alpha_1 = a \mid b$, $\alpha_2 = a \mid c$, $w = a$
   $\Longrightarrow$ unique decomposition (a) but two analyses (T_1) and (T_2)

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Goal: make both unique $\Longrightarrow$ deterministic scanning

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Ensuring Uniqueness

Two principles:

1. Principle of the longest match ("maximal munch tokenization")
   - for uniqueness of decomposition
   - make lexemes as long as possible
   - motivated by applications: e.g., every (non-empty) prefix of an identifier is also an identifier

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1. Principle of the longest match ("maximal munch tokenization")
   - for uniqueness of decomposition
   - make lexemes as long as possible
   - motivated by applications: e.g., every (non-empty) prefix of an identifier is also an identifier
2. Principle of the first match
   - for uniqueness of analysis
   - choose first matching regular expression (in the given order)
   - therefore: arrange keywords before identifiers (if keywords protected)

Principle of the Longest Match

Definition 3.5 (Longest-match decomposition)

A decomposition $(w_1, \ldots, w_k)$ of $w \in \Omega^+$ w.r.t. $\alpha_1, \ldots, \alpha_n \in RE_\Omega$ is called a longest-match decomposition (LM decomposition) if, for every $i \in [k]$, $x \in \Omega^+$, and $y \in \Omega^*$,

$$w = w_1 \ldots w_i x y \implies \text{there is no } j \in [n] \text{ such that } w_i x \in [\![\alpha_j]\!]$$

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Corollary 3.6

Given w and $\alpha_1, \ldots, \alpha_n$,

- *at most one LM decomposition of w exists (clear by definition) and*

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Corollary 3.6

Given w and $\alpha_1, \ldots, \alpha_n$,

- *at most one LM decomposition of w exists (clear by definition) and*
- *it is possible that w has a decomposition but no LM decomposition (see following example).*

Example 3.7

$w = aab$, $\alpha_1 = a^+$, $\alpha_2 = ab$

$\implies$ (a, ab) is a decomposition but no LM decomposition exists

Principle of the First Match

Problem: a (unique) LM decomposition can have several associated analyses (since $[\![\alpha_i]\!] \cap [\![\alpha_j]\!] \neq \emptyset$ with $i \neq j$ is possible; cf. keyword/identifier problem)

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Definition 3.8 (First-longest-match analysis)

Let $(w_1, \ldots, w_k)$ be the LM decomposition of $w \in \Omega^+$ w.r.t. $\alpha_1, \ldots, \alpha_n \in RE_\Omega$. Its first-longest-match analysis (FLM analysis) $(T_{i_1}, \ldots, T_{i_k})$ is determined by

$$i_j := \min\{l \in [n] \mid w_j \in [\![\alpha_l]\!]\}$$

for every $j \in [k]$.

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Corollary 3.9

Given w and $\alpha_1, \ldots, \alpha_n$, there is at most one FLM analysis of w. It exists iff the LM decomposition of w exists.

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Algorithm 3.10 (FLM analysis – overview)

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Procedure:

1. *for every* $i \in [n]$, *construct* $\mathfrak{A}_i \in DFA_\Omega$ *such that* $L(\mathfrak{A}_i) = [\![\alpha_i]\!]$ *(see DFA method; Algorithm 2.9)*

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Output: *FLM analysis of* w *(if existing)*

(2) The Product Automaton

Definition 3.11 (Product automaton)

Let $\mathfrak{A}_i = \langle Q_i, \Omega, \delta_i, q_0^{(i)}, F_i \rangle \in DFA_\Omega$ for every $i \in [n]$. The product automaton $\mathfrak{A} = \langle Q, \Omega, \delta, q_0, F \rangle \in DFA_\Omega$ is defined by

- $Q := Q_1 \times \ldots \times Q_n$
- $q_0 := (q_0^{(1)}, \ldots, q_0^{(n)})$
- $\delta((q^{(1)}, \ldots, q^{(n)}), a) := (\delta_1(q^{(1)}, a), \ldots, \delta_n(q^{(n)}, a))$
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Lemma 3.12

The above construction yields $L(\mathfrak{A}) = \bigcup_{i=1}^{n} L(\mathfrak{A}_i)$ $(= \bigcup_{i=1}^{n} [\![\alpha_i]\!])$.

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The above construction yields $L(\mathfrak{A}) = \bigcup_{i=1}^{n} L(\mathfrak{A}_i)$ $(= \bigcup_{i=1}^{n} [\![\alpha_i]\!])$.

Remark: similar construction for intersection ($F := F_1 \times \ldots \times F_n$)

(3) Partitioning the Final States

Definition 3.13 (Partitioning of final states)

Let $\mathfrak{A} = \langle Q, \Omega, \delta, q_0, F \rangle \in DFA_\Omega$ be the product automaton as constructed before. Its set of final states is partitioned into $F = \biguplus_{i=1}^{n} F^{(i)}$ by the requirement

$$(q^{(1)}, \ldots, q^{(n)}) \in F^{(i)} \iff q^{(i)} \in F_i \text{ and } \forall j \in [i-1] : q^{(j)} \notin F_j$$

(or: $F^{(i)} := (Q_1 \setminus F_1) \times \ldots \times (Q_{i-1} \setminus F_{i-1}) \times F_i \times Q_{i+1} \times \ldots \times Q_n$)

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Corollary 3.14

The above construction yields ($w \in \Omega^+$, $i \in [n]$):

$$\delta^*(q_0, w) \in F^{(i)} \textit{ iff } w \in [\![\alpha_i]\!] \textit{ and } w \notin \bigcup_{j=1}^{i-1} [\![\alpha_j]\!].$$

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Definition 3.15 (Productive states)

Given $\mathfrak{A}$ as above, $q \in Q$ is called productive if there exists $w \in \Omega^*$ such that $\delta^*(q, w) \in F$. Notation: productive states $P \subseteq Q$ (thus $F \subseteq P$).

(4) The Backtracking DFA I

Goal: extend $\mathfrak{A}$ to the backtracking DFA $\mathfrak{B}$ with output by equipping the input tape with two pointers: a backtracking head for marking the last encountered match, and a lookahead for determining the longest match.

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A configuration of $\mathfrak{B}$ has three components
(remember: $\Sigma := \{T_1, \ldots, T_n\}$ denotes the set of tokens):

1. a mode $m \in \{N\} \uplus \Sigma$:
   - $m = N$ ("normal"): look for initial match (no final state reached yet)
   - $m = T \in \Sigma$: token T has been recognized, looking for possible longer match

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2. an input tape $vqw \in \Omega^* \cdot Q \cdot \Omega^*$:
   - v: lookahead part of input ($v \neq \varepsilon \implies m \in \Sigma$)
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   - q: current state of $\mathfrak{A}$
   - w: remaining input
3. an output tape $W \in \Sigma^* \cdot \{\varepsilon, \text{lexerr}\}$:
   - Σ^*: sequence of tokens recognized so far
   - lexerr: a lexical error has occurred (i.e., a non-productive state was entered or the suffix of the input is not a valid lexeme)

Definition 3.16 (Backtracking DFA)

- The set of configurations of $\mathfrak{B}$ is given by
$$(\{N\} \uplus \Sigma) \times \Omega^* \cdot Q \cdot \Omega^* \times \Sigma^* \cdot \{\varepsilon, \text{lexerr}\}$$
- The initial configuration for an input word $w \in \Omega^+$ is $(N, q_0 w, \varepsilon)$.

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  - normal mode: look for initial match

$$(N, qaw, W) \vdash \begin{cases} (T_i, q'w, W) & \text{if } q' \in F^{(i)} \\ (N, q'w, W) & \text{if } q' \in P \setminus F \\ \textbf{output: } W \cdot \mathrm{lexerr} & \text{if } q' \notin P \end{cases}$$

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  - backtrack mode: look for longest match
$$(T, vqaw, W) \vdash \begin{cases} (T_i, q'w, W) & \text{if } q' \in F^{(i)} \\ (T, vaq'w, W) & \text{if } q' \in P \setminus F \\ (N, q_0 vaw, WT) & \text{if } q' \notin P \end{cases}$$

(4) The Backtracking DFA II

Definition 3.16 (Backtracking DFA)

- The set of configurations of $\mathfrak{B}$ is given by
$$(\{N\} \uplus \Sigma) \times \Omega^* \cdot Q \cdot \Omega^* \times \Sigma^* \cdot \{\varepsilon, \text{lexerr}\}$$
- The initial configuration for an input word $w \in \Omega^+$ is $(N, q_0 w, \varepsilon)$.
- The transitions of $\mathfrak{B}$ are defined as follows (where $q' := \delta(q, a)$):
  - normal mode: look for initial match
$$(N, qaw, W) \vdash \begin{cases} (T_i, q'w, W) & \text{if } q' \in F^{(i)} \\ (N, q'w, W) & \text{if } q' \in P \setminus F \\ \textbf{output: } W \cdot \text{lexerr} & \text{if } q' \notin P \end{cases}$$
  - backtrack mode: look for longest match
$$(T, vqaw, W) \vdash \begin{cases} (T_i, q'w, W) & \text{if } q' \in F^{(i)} \\ (T, vaq'w, W) & \text{if } q' \in P \setminus F \\ (N, q_0 vaw, WT) & \text{if } q' \notin P \end{cases}$$
  - end of input
$$\begin{aligned} (T, q, W) &\vdash \textbf{output: } WT && \text{if } q \in F \\ (N, q, W) &\vdash \textbf{output: } W \cdot \text{lexerr} && \text{if } q \in P \setminus F \\ (T, vaq, W) &\vdash (N, q_0 va, WT) && \text{if } q \in P \setminus F \end{aligned}$$

(4) The Backtracking DFA III

Lemma 3.17

Given the backtracking DFA $\mathfrak{B}$ *as before and* $w \in \Omega^+$,

$$(N, q_0 w, \varepsilon) \quad \vdash^* \quad \begin{cases} W \in \Sigma^* & \textit{iff } W \textit{ is the FLM analysis of } w \\ W \cdot \mathrm{lexerr} & \textit{iff no FLM analysis of } w \textit{ exists} \end{cases}$$

Proof.

(omitted) □

Lemma 3.17

Given the backtracking DFA $\mathfrak{B}$ as before and $w \in \Omega^+$,

$$(N, q_0 w, \varepsilon) \quad \vdash^* \quad \begin{cases} W \in \Sigma^* & \textit{iff } W \textit{ is the FLM analysis of } w \\ W \cdot \mathrm{lexerr} & \textit{iff no FLM analysis of } w \textit{ exists} \end{cases}$$

Proof.

(omitted) □

Example 3.18

- $\Omega = \{a, b\}$, $w = aaba$
- $n = 3$, $\Sigma = \{T_1, T_2.T_3\}$
- $\alpha_1 = a$ (“keyword”), $\alpha_2 = a^+$ (“identifier”), $\alpha_3 = b$ (“operator”)

(on the board)

(4) The Backtracking DFA IV

Remarks:

- Time complexity: $\mathcal{O}(|w|^2)$ in worst case

Example 3.19

$\alpha_1 = a$, $\alpha_2 = a^*b$, $w = a^m$ requires $\mathcal{O}(m^2)$

(4) The Backtracking DFA IV

Remarks:

- Time complexity: $\mathcal{O}(|w|^2)$ in worst case

Example 3.19

$\alpha_1 = a$, $\alpha_2 = a^*b$, $w = a^m$ requires $\mathcal{O}(m^2)$

- Improvement by tabular method (similar to Knuth-Morris-Pratt Algorithm for pattern matching in strings)

  Literature: Th. Reps: *"Maximal-Munch" Tokenization in Linear Time*, ACM TOPLAS 20(2), 1998, 259–273