# Overview

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| |
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*goal*: computation of $S/\!\approx_{\mathcal{T}}$

stutter bisimulation equivalence classes
without divergence

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- refinement relies on a $Pre^*_{\mathcal{B}}$-operator

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$\mathsf{Pre}^*_{\mathcal{B}}(C)$ = set of states $s_0$ that have a path fragment

$$s_0\, s_1\, \ldots\, s_{m-1}\, s_m$$

such that $m \geq 0$ and $s_0, s_1, \ldots, s_{m-1}$ belong to the same block of $\mathcal{B}$ and $s_m \in C$

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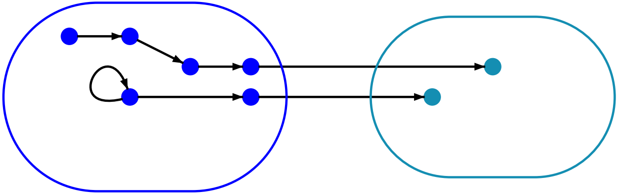

block $B \in \mathcal{B}$ block $C \in \mathcal{B}$

$B \neq C$

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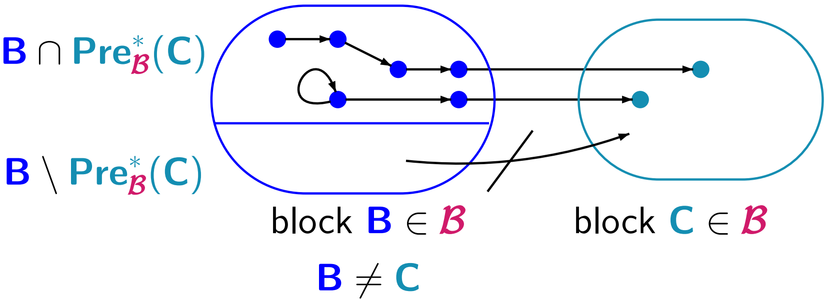

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$\mathcal{B}$ is finer than $\mathcal{B}_{AP}$ and coarser than $S/{\approx_{\mathcal{T}}}$

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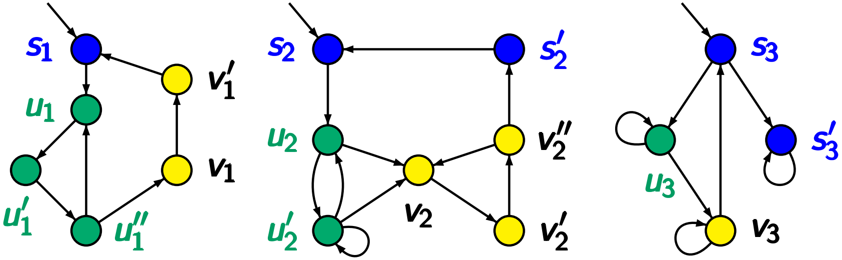
- search for splitter pair in time $\mathcal{O}(m)$
- complexity: $\mathcal{O}(|S| \cdot |AP| + |S| \cdot m)$

where $m$ = number of edges $\geq |S|$

# Example: computation of $S/{\approx_{\mathcal{T}}}$





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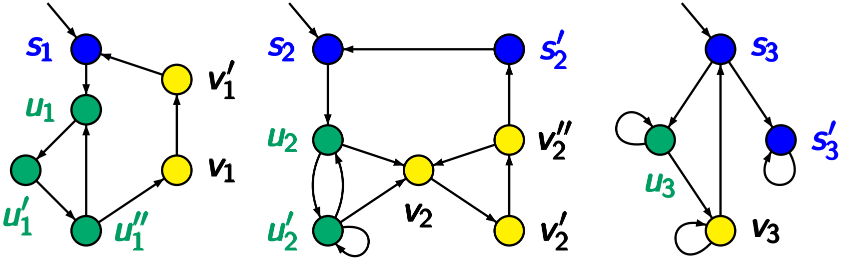


initial partition: $\{s_1, s_2, s_2', s_3, s_3'\}$,
$\{u_1, u_1', u_1'', u_2, u_2', u_3\}$,
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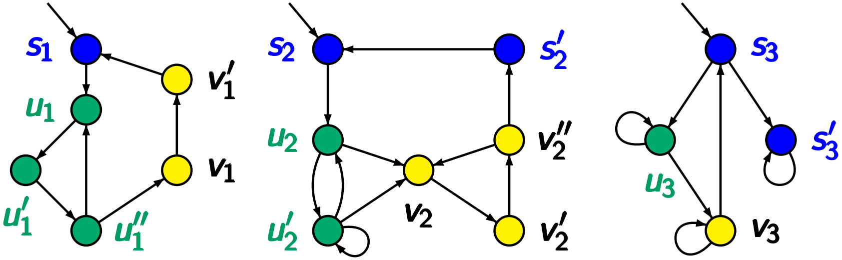


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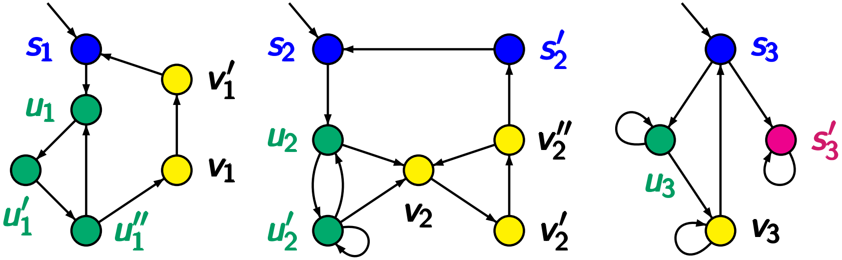


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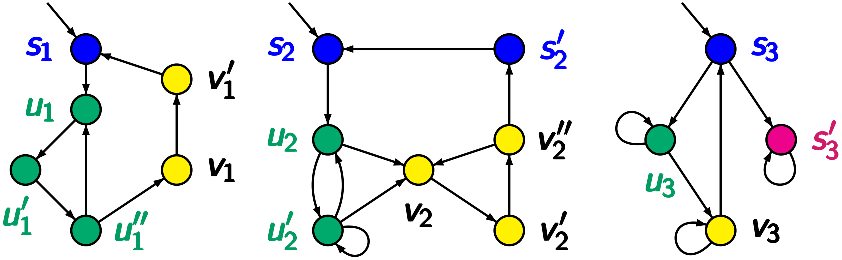


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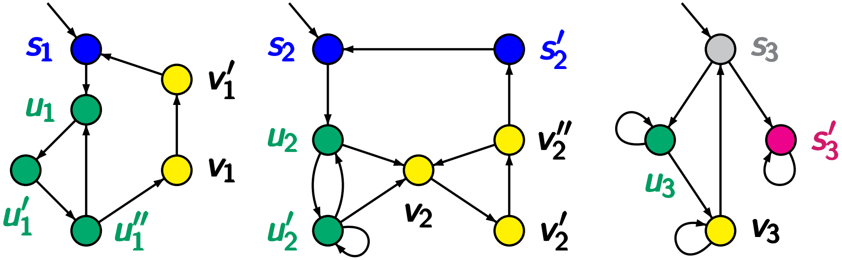


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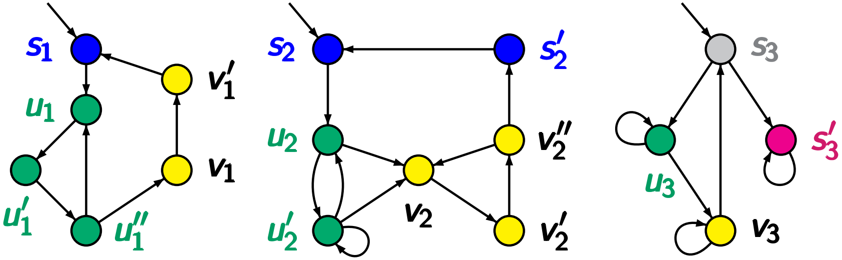


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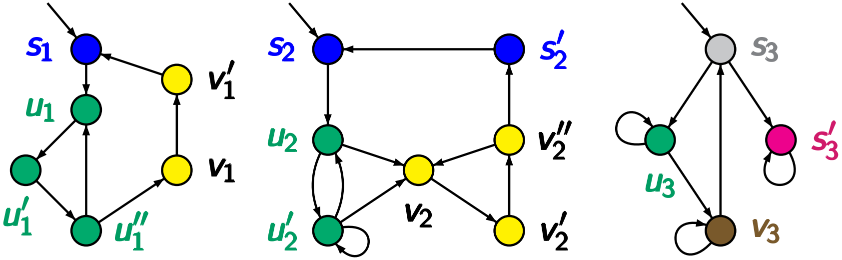


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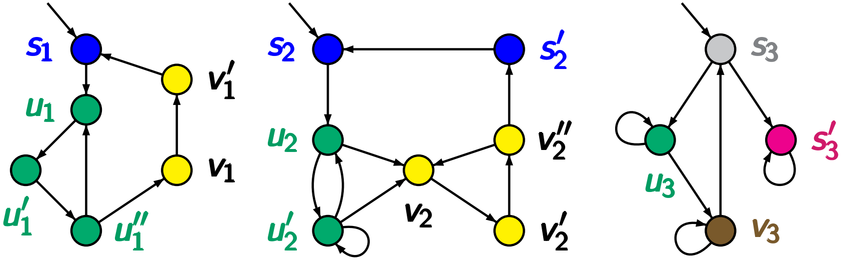


splitter pair $(B_1, B_2)$:

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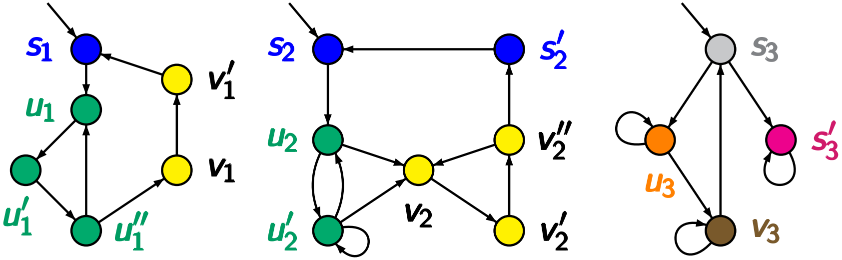
# Example: computation of $S/\!\approx_{\mathcal{T}}$



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3. derive $S/\approx^{\mathsf{div}}_{\mathcal{T}}$ from $\overline{S}/\approx_{\overline{\mathcal{T}}}$

# Stutter cycles

STUTTER5.4-63

# Stutter cycles



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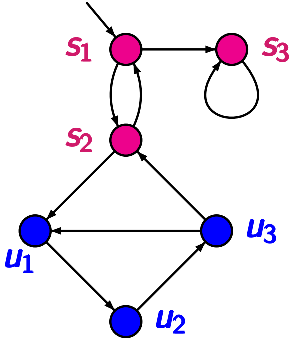

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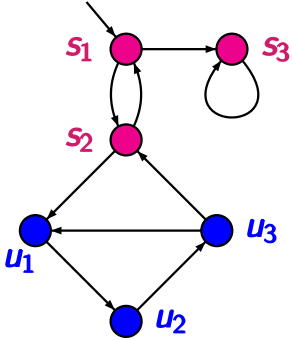


stutter cycles, e.g.:

$s_1\, s_2\, s_1$

$s_3\, s_3$

$u_1\, u_2\, u_3\, u_1$

A stutter cycle is a cycle $s_0\, s_1\, \ldots\, s_n$ in $\mathcal{T}$ s.t.
$$L(s_0) = L(s_i) \text{ for } i = 1, \ldots, n.$$

If $s_0\, s_1\, \ldots\, s_n$ is a stutter cycle then
$$s_0 \approx_{\mathcal{T}}^{\mathsf{div}} s_1 \approx_{\mathcal{T}}^{\mathsf{div}} \ldots \approx_{\mathcal{T}}^{\mathsf{div}} s_n$$

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*Proof:* show that

$$\mathcal{R} = \mathbf{id} \cup \{(s_i, s_j) : i, j{=}1, \ldots, n\}$$

is a divergence-sensitive stutter bisimulation.

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If state $s$ is $\approx_{\mathcal{T}}^{\mathsf{div}}$-divergent then $s$ belongs to a stutter cycle.

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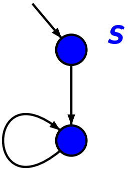

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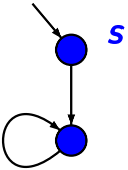


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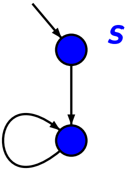


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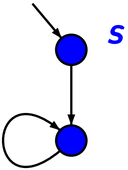
> If state $s$ is $\approx^{\mathsf{div}}_{\mathcal{T}}$-divergent then $s$ can reach a stutter cycle by stutter steps within $[s]$.

Let $\mathcal{T}$ be a finite transition system.

If state $s$ is $\approx^{\text{div}}_{\mathcal{T}}$-divergent then $s$ belongs to a stutter cycle.

**wrong.**

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If $s \longrightarrow s'$ is a stutter step, i.e., $L(s) = L(s')$, then $s \approx^{\mathsf{div}}_{\mathcal{T}} s'$.

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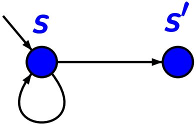

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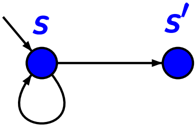


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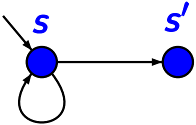


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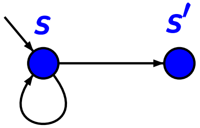


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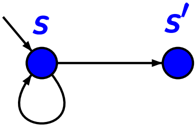


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1. If $s \to s'$ is a stutter step and $Post(s) = \{s'\}$ then $s \approx_{\mathcal{T}}^{\text{div}} s'$.
2. If $s$ and $s'$ are on the same stutter cycle then $s \approx_{\mathcal{T}}^{\text{div}} s'$.

# Divergence-sensitive expansion

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| | |
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| stutter bis. equiv. with divergence in $\mathcal{T}$ | stutter bis. equiv. without divergence in $\overline{\mathcal{T}}$ |

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where ***div*** is a new state, not contained in $S$

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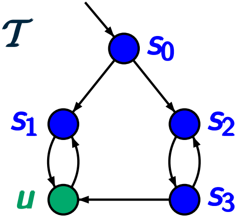
- actions names: irrelevant

# Example: divergence-sensitive expansion



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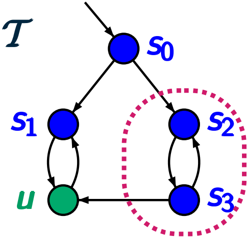
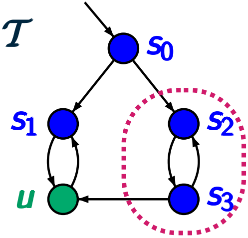



$AP = \{a\}$

stutter cycle $s_2\, s_3\, s_2$

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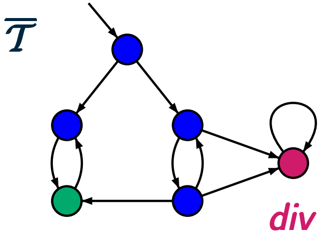

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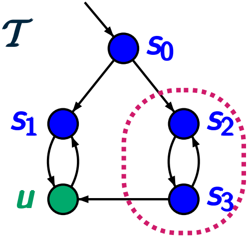


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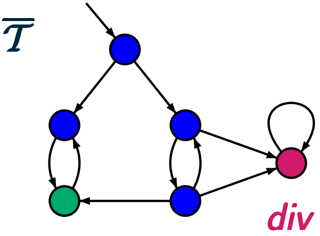


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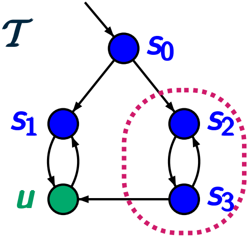


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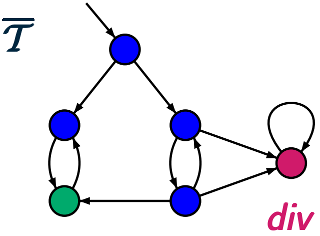


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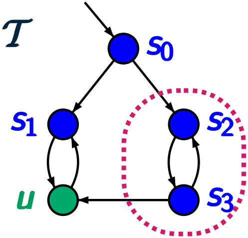


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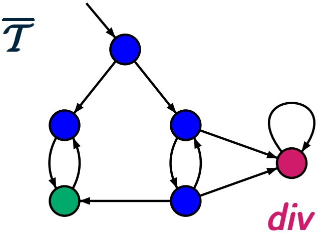


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# $\mathcal{T}$ and its divergence-sensitive expansion $\overline{\mathcal{T}}$ stutter5.4-67a

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(3) $s_1$ is $\mathcal{R}$-divergent iff $s_2$ is $\mathcal{R}$-divergent

## $\approx_{\mathcal{T}}$ is finer than $\approx_{\mathcal{T}}^{\mathsf{div}}$



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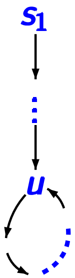


$\mathcal{R}$-divergent path from $s_1$

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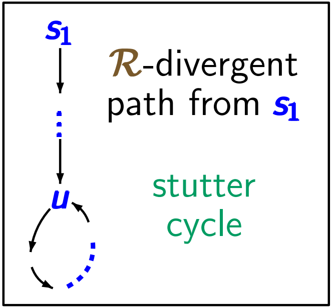

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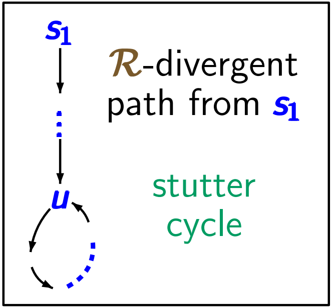


$u \approx_{\mathcal{T}} s_1 \approx_{\mathcal{T}} s_2$

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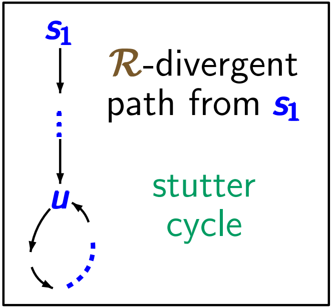


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$\downarrow$

*div*

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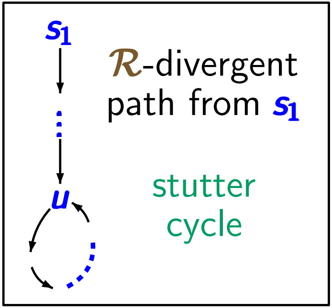
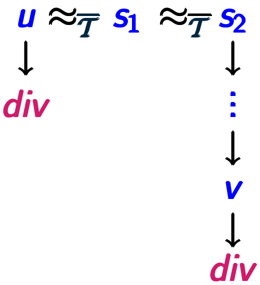
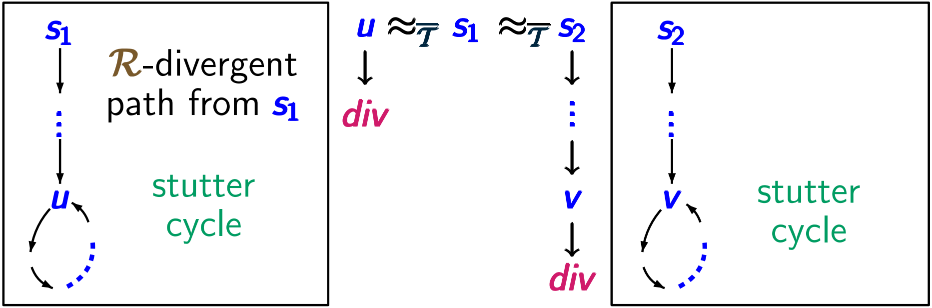


Claim: $\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}} s_2 \}$ is a divergence-sensitive stutter bisimulation for $\mathcal{T}$.

Proof of the divergence-sensitivity of $\mathcal{R}$:

suppose $s_1 \approx_{\mathcal{T}} s_2$, $s_1$ $\mathcal{R}$-divergent



# $\approx_{\mathcal{T}}$ is finer than $\approx_{\mathcal{T}}^{\mathrm{div}}$



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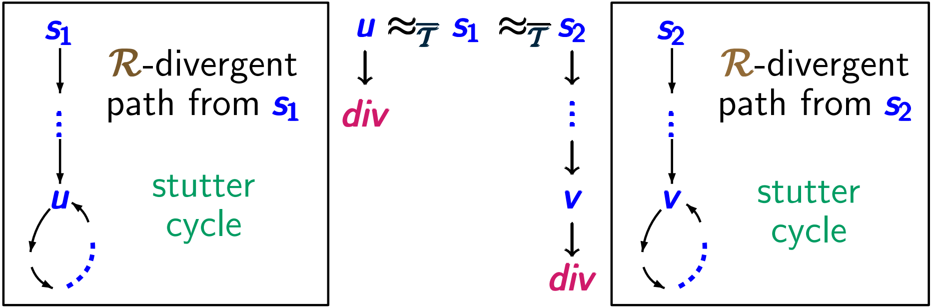


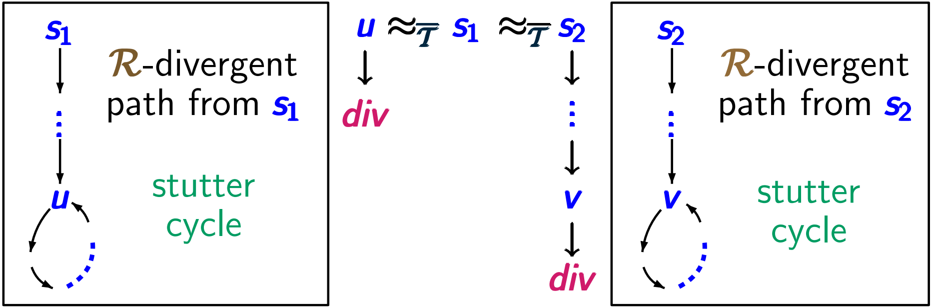
# $\approx_{\mathcal{T}}$ is finer than $\approx_{\mathcal{T}}^{\mathsf{div}}$



Claim: $\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}} s_2 \}$ is a divergence-sensitive stutter bisimulation for $\mathcal{T}$.

Proof of the divergence-sensitivity of $\mathcal{R}$:

suppose $s_1 \approx_{\mathcal{T}} s_2$, $s_1$ $\mathcal{R}$-divergent $\Longrightarrow$ $s_2$ $\mathcal{R}$-divergent



# $\approx_{\mathcal{T}}^{\text{div}}$ is finer than $\approx_{\mathcal{T}}$

$$\mathcal{R} = \left\{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \right\} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

# $\approx_{\mathcal{T}}^{\mathsf{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



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*Proof:*

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*Proof:*

(1) labeling condition: $\surd$

# $\approx_{\mathcal{T}}^{\mathbf{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$

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is a stutter bisimulation for $\overline{\mathcal{T}}$.

*Proof:*

(1) labeling condition: $\checkmark$

(2) simulation condition:

$$\mathcal{R} = \left\{ (s_1, s_2) \in S \times S : s_1 \approx^{\text{div}}_{\mathcal{T}} s_2 \right\} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

*Proof:*

(1) labeling condition: $\checkmark$

(2) simulation condition:

- for the pair $(div, div) \in \mathcal{R}$: $\checkmark$

# $\approx_{\mathcal{T}}^{\mathsf{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \left\{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \right\} \cup \{(div, div)\}$$
is a stutter bisimulation for $\overline{\mathcal{T}}$.

*Proof:*

(1) labeling condition: $\surd$

(2) simulation condition:

- for the pair $(div, div) \in \mathcal{R}$: $\surd$
- regard now a pair $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

# $\approx_{\mathcal{T}}^{\text{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\text{div}} s_2 \} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

# $\approx_{\mathcal{T}}^{\mathbf{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\mathbf{div}} s_2 \} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

show that:

$$\begin{tikzcd} s_1 \arrow[r, dash, "\mathcal{R}"] \arrow[d] & s_2 \\ s_1' & \end{tikzcd}$$

can be completed to

$$\begin{tikzcd} s_1 \arrow[r, dash, "\mathcal{R}"] \arrow[dd] & s_2 \arrow[d, dotted] \\ & u_n \arrow[d] \\ s_1' \arrow[r, dash, "\mathcal{R}"] & s_2' \end{tikzcd}$$

$\mathcal{R}$

with $(s_1', s_1) \notin \mathcal{R}$

# $\approx^{\mathrm{div}}_{\mathcal{T}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx^{\mathrm{div}}_{\mathcal{T}} s_2 \} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $\boxed{s_1, s_2 \in S}$

show that:

$$\begin{tikzcd}
s_1 \arrow[d] & \approx^{\mathrm{div}}_{\mathcal{T}} & s_2 \\
s_1' & &
\end{tikzcd}$$

with $(s_1', s_1) \notin \mathcal{R}$

can be completed to

$$\begin{tikzcd}
s_1 \arrow[ddd] & \approx^{\mathrm{div}}_{\mathcal{T}} & s_2 \arrow[d] \arrow[dd, bend left, dash, "\approx^{\mathrm{div}}_{\mathcal{T}}"] \\
& & \vdots \arrow[d] \\
& & u_n \arrow[d] \\
s_1' & -\mathcal{R}- & s_2'
\end{tikzcd}$$

# $\approx^{\text{div}}_{\mathcal{T}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx^{\text{div}}_{\mathcal{T}} s_2 \} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

show that:

$$\begin{array}{ccc} s_1 & \approx^{\text{div}}_{\mathcal{T}} & s_2 \\ \downarrow & & \\ s_1' & & \end{array}$$

can be completed to

$$\begin{tikzcd} s_1 \arrow[ddd] & \approx^{\text{div}}_{\mathcal{T}} & s_2 \arrow[d] \arrow[dd, phantom, bend left=60, "\approx^{\text{div}}_{\mathcal{T}}"] \\ & & \vdots \arrow[d] \\ & & u_n \arrow[d] \\ s_1' & -\mathcal{R}- & s_2' \end{tikzcd}$$

with $(s_1', s_1) \notin \mathcal{R}$

$\boxed{\text{obvious if } s_1' \in S}$

# $\approx_{\mathcal{T}}^{\mathsf{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \} \cup \{(div, div)\}$
is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

show that:

$$\begin{tikzcd} s_1 \arrow[d] & \approx_{\mathcal{T}}^{\mathsf{div}} & s_2 \\ s_1' & & \end{tikzcd}$$

with $(s_1', s_1) \notin \mathcal{R}$

$\boxed{\text{consider } s_1' = div}$

can be completed to

$$\begin{tikzcd} s_1 \arrow[ddd] & \approx_{\mathcal{T}}^{\mathsf{div}} & s_2 \arrow[d] & \\ & & \vdots \arrow[d] & \approx_{\mathcal{T}}^{\mathsf{div}} \\ & & u_n \arrow[d] & \\ s_1' & -\mathcal{R}- & s_2' & \end{tikzcd}$$

# $\approx_{\mathcal{T}}^{\mathsf{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

$$\begin{tikzcd}
s_1 \arrow[d] & \approx_{\mathcal{T}}^{\mathsf{div}} & s_2 \\
div & &
\end{tikzcd}$$

# $\approx_{\mathcal{T}}^{\mathsf{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \left\{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \right\} \cup \{(div, div)\}$$

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simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

$$\begin{tikzcd}
s_1 \arrow[d] & \approx_{\mathcal{T}}^{\mathsf{div}} & s_2 \\
div & &
\end{tikzcd}$$

$s_1$ is on stutter cycle

# $\approx^{\mathsf{div}}_{\mathcal{T}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \left\{ (s_1, s_2) \in S \times S : s_1 \approx^{\mathsf{div}}_{\mathcal{T}} s_2 \right\} \cup \{(div, div)\}$$
is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

$$\begin{tikzcd} s_1 \arrow[d] & \approx^{\mathsf{div}}_{\mathcal{T}} & s_2 \\ div & & \end{tikzcd}$$

$s_1$ is on stutter cycle

$$\Downarrow$$

$s_1, s_2$ $\approx^{\mathsf{div}}_{\mathcal{T}}$-divergent

$$\mathcal{R} = \left\{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\mathbf{div}} s_2 \right\} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

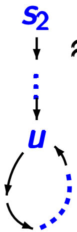
simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

$$\begin{array}{ccc} s_1 & \approx_{\mathcal{T}}^{\mathbf{div}} & s_2 \\ \downarrow & & \\ div & & \end{array}$$

$s_1$ is on stutter cycle

$\Downarrow$

$s_1, s_2$ $\approx_{\mathcal{T}}^{\mathbf{div}}$-divergent




$\approx_{\mathcal{T}}^{\mathbf{div}}$-divergent path from $s_2$

# $\approx_{\mathcal{T}}^{\text{div}}$ is finer than $\approx_{\overline{\mathcal{T}}}$



$$\mathcal{R} = \left\{ (s_1, s_2) \in S \times S : s_1 \approx_{\mathcal{T}}^{\text{div}} s_2 \right\} \cup \{(div, div)\}$$

is a stutter bisimulation for $\overline{\mathcal{T}}$.

simulation condition for $(s_1, s_2) \in \mathcal{R}$ where $s_1, s_2 \in S$

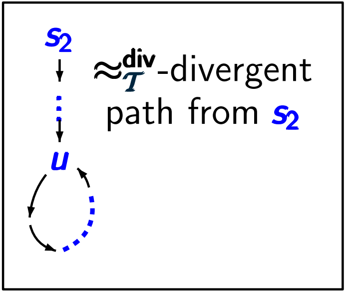
$s_1 \approx_{\mathcal{T}}^{\text{div}} s_2$

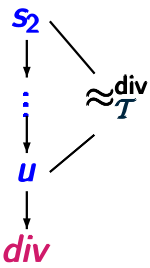
$s_1 \to div$

$s_1$ is on stutter cycle

$\Downarrow$

$s_1, s_2$ $\approx_{\mathcal{T}}^{\text{div}}$-divergent



# $\approx_{\mathcal{T}}^{\mathsf{div}}$ and $\approx_{\overline{\mathcal{T}}}$



For finite transition systems:

$$s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \quad \text{iff} \quad s_1 \approx_{\overline{\mathcal{T}}} s_2$$

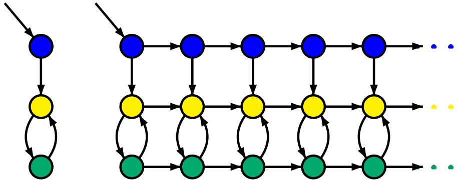
# $\approx_{\mathcal{T}}^{\mathsf{div}}$ and $\approx_{\mathcal{T}}$ for infinite systems

For finite transition systems:

$$s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \quad \text{iff} \quad s_1 \approx_{\mathcal{T}} s_2$$

**wrong** for infinite transition systems

$\mathcal{T}$



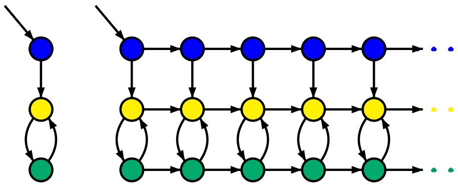
# $\approx_{\mathcal{T}}^{\mathsf{div}}$ and $\approx_{\overline{\mathcal{T}}}$ for infinite systems

For finite transition systems:

$$s_1 \approx_{\mathcal{T}}^{\mathsf{div}} s_2 \quad \text{iff} \quad s_1 \approx_{\overline{\mathcal{T}}} s_2$$

**wrong** for infinite transition systems

$\mathcal{T} \mathrel{\widehat{=}} \overline{\mathcal{T}}$



As there are no stutter cycles:

$\mathcal{T}$ agrees with the reachable part of $\overline{\mathcal{T}}$

# $\approx^{\mathsf{div}}_{\mathcal{T}}$ and $\approx_{\overline{\mathcal{T}}}$ for infinite systems

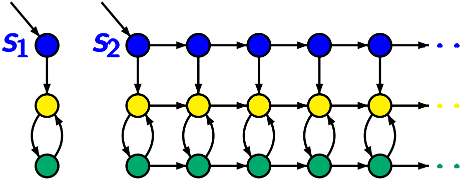


For finite transition systems:

$$s_1 \approx^{\mathsf{div}}_{\mathcal{T}} s_2 \quad \text{iff} \quad s_1 \approx_{\overline{\mathcal{T}}} s_2$$

**wrong** for infinite transition systems

$\mathcal{T} \mathrel{\widehat{=}} \overline{\mathcal{T}}$




As there are no stutter cycles:

$\mathcal{T}$ agrees with the reachable part of $\overline{\mathcal{T}}$

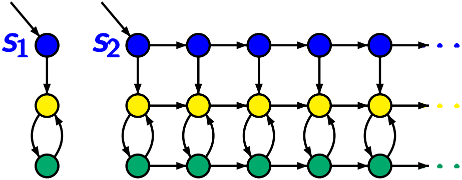
$s_1 \approx s_2$,

For finite transition systems:

$$s_1 \approx^{\mathsf{div}}_{\mathcal{T}} s_2 \quad \text{iff} \quad s_1 \approx_{\overline{\mathcal{T}}} s_2$$

**wrong** for infinite transition systems

$\mathcal{T} \,\widehat{=}\, \overline{\mathcal{T}}$




As there are no stutter cycles:

$\mathcal{T}$ agrees with the reachable part of $\overline{\mathcal{T}}$

$s_1 \approx s_2$, but $s_1 \not\approx^{\mathsf{div}} s_2$

# Computing $\mathcal{T}/\approx^{\mathsf{div}}$ for finite TS

# Computing $\mathcal{T}/{\approx^{\mathsf{div}}}$ for finite TS



| transition system $\mathcal{T}$ |
| --- |

atomic prop.: $AP$

# Computing $\mathcal{T}/{\approx^{\mathsf{div}}}$ for finite TS



transition system $\mathcal{T}$

$\downarrow$

divergence-sensitive expansion $\overline{\mathcal{T}}$

atomic prop.: $AP$

# Computing $\mathcal{T}/\approx^{\textsf{div}}$ for finite TS



$$\boxed{\text{transition system } \mathcal{T}} \longrightarrow \boxed{\text{divergence-sensitive expansion } \overline{\mathcal{T}}}$$

atomic prop.: $AP$

add trap state $div$

# Computing $\mathcal{T}/{\approx^{\mathsf{div}}}$ for finite TS



| | | |
|---|---|---|
| transition system $\mathcal{T}$ | $\downarrow$ | atomic prop.: $AP$ |
| divergence-sensitive expansion $\overline{\mathcal{T}}$ | | add trap state $div$<br>atomic prop.: $AP \cup \{div\}$ |

# Computing $\mathcal{T}/{\approx^{\mathbf{div}}}$ for finite TS



transition system $\mathcal{T}$

$\downarrow$

divergence-sensitive expansion $\overline{\mathcal{T}}$

$\downarrow$

quotient system $\overline{\mathcal{T}}/{\approx}$ w.r.t. stutter bisim.

atomic prop.: $AP$

add trap state $div$

atomic prop.: $AP \cup \{div\}$

# Computing $\mathcal{T}/\approx^{\text{div}}$ for finite TS



transition system $\mathcal{T}$

atomic prop.: $AP$

$\downarrow$

divergence-sensitive expansion $\overline{\mathcal{T}}$

add trap state $div$

atomic prop.: $AP \cup \{div\}$

$\downarrow$

quotient system $\overline{\mathcal{T}}/\approx$ w.r.t. stutter bisim.

$\downarrow$

quotient system $\mathcal{T}/\approx^{\text{div}}$ w.r.t. stutter bisim. with divergence

# Computing $\mathcal{T}/\approx^{\mathsf{div}}$ for finite TS



$$\boxed{\text{transition system } \mathcal{T}}$$
$$\downarrow$$
$$\boxed{\text{divergence-sensitive expansion } \overline{\mathcal{T}}}$$
$$\downarrow$$
$$\boxed{\text{quotient system } \overline{\mathcal{T}}/\approx \text{ w.r.t. stutter bisim.}}$$
$$\downarrow$$
$$\boxed{\text{quotient system } \mathcal{T}/\approx^{\mathsf{div}} \text{ w.r.t. stutter bisim. with divergence}}$$

atomic prop.: $AP$

add trap state *div*

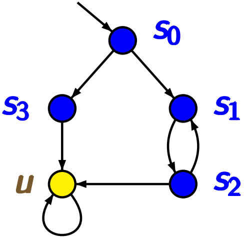
atomic prop.: $AP \cup \{div\}$

redirect $[s] \rightarrow div$

to $[s] \rightarrow [s]$

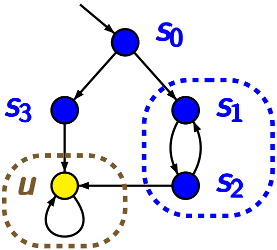
# Example: computing $\mathcal{T}/\approx^{\text{div}}$






$AP = \{blue\}$

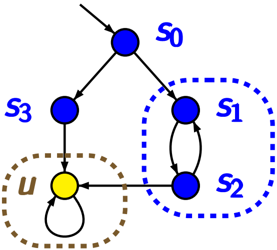
# Example: computing $\mathcal{T}/\approx^{\mathsf{div}}$



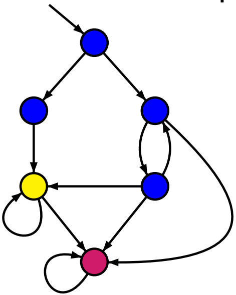

$AP = \{blue\}$

stutter cycles

$u\,u$ and $s_1\,s_2\,s_1$




div.-sensitive exp. $\overline{\mathcal{T}}$



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stutter cycles

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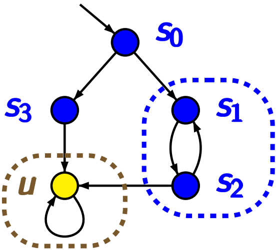
# Example: computing $\mathcal{T}/\approx^{\text{div}}$

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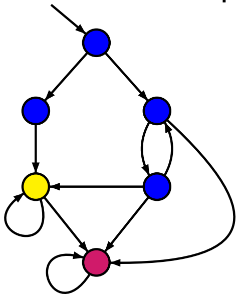
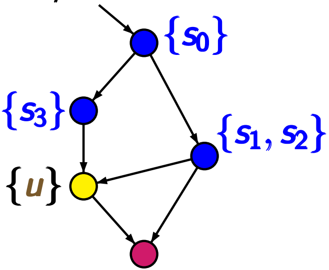
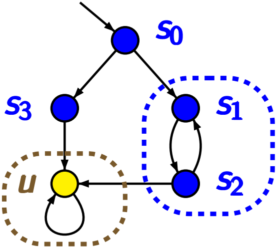
div.-sensitive exp. $\overline{\mathcal{T}}$



$\overline{\mathcal{T}}/\approx$



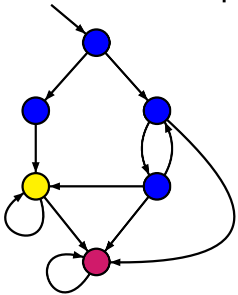



$AP = \{\mathit{blue}\}$

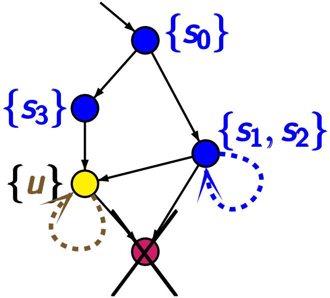
stutter cycles

$u\,u$ and $s_1\,s_2\,s_1$

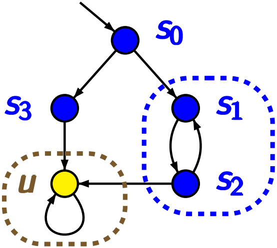
div.-sensitive exp. $\overline{\mathcal{T}}$



$\overline{\mathcal{T}}/\!\approx \;\rightsquigarrow\; \mathcal{T}/\!\approx^{\text{div}}$



# Example: computing $\mathcal{T}/{\approx^{\mathsf{div}}}$

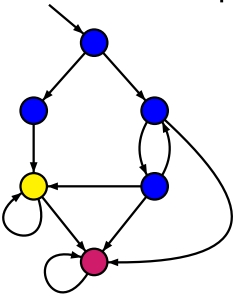



$AP = \{ \textit{blue} \}$

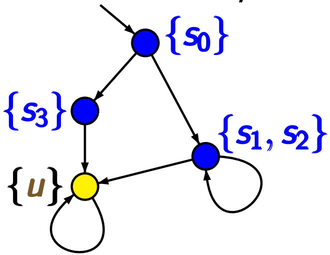
stutter cycles

$u\,u$ and $s_1\,s_2\,s_1$

div.-sensitive exp. $\overline{\mathcal{T}}$



$\mathcal{T}/{\approx^{\mathsf{div}}}$



# Summary: equivalences

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| | bisimulation equivalence | stutter bisimulation with divergence | trace equivalence |
|---|---|---|---|
| temporal logic characteriz. | **CTL*** <br> **CTL** | $\mathbf{CTL^*}_{\setminus \bigcirc}$ <br> $\mathbf{CTL}_{\setminus \bigcirc}$ | **LTL** |
| | | | |

# Summary: equivalences



| | bisimulation equivalence | stutter bisimulation with divergence | trace equivalence |
|---|---|---|---|
| temporal logic characteriz. | CTL*<br>CTL | $\text{CTL}^*_{\setminus \bigcirc}$<br>$\text{CTL}_{\setminus \bigcirc}$ | LTL |
| equivalence checking | PTIME | PTIME | PSPACE-complete |

# Summary: equivalences



| | bisimulation equivalence | stutter bisimulation with divergence | trace equivalence |
|---|---|---|---|
| temporal logic characteriz. | CTL* <br> CTL | $\text{CTL}^*_{\setminus \bigcirc}$ <br> $\text{CTL}_{\setminus \bigcirc}$ | LTL |
| equivalence checking | PTIME <br> $\mathcal{O}(m \cdot \log \lvert S \rvert)$ | PTIME <br> $\mathcal{O}(m \cdot \log \lvert S \rvert)$ | PSPACE-complete |

where $AP$ is fixed and

$m$ = number of edges $\geq$ number of states = $|S|$

# Summary: equivalences



| | bisimulation equivalence | stutter bisimulation with divergence | trace equivalence |
|---|---|---|---|
| temporal logic characteriz. | CTL* <br> CTL | $CTL^*_{\setminus \bigcirc}$ <br> $CTL_{\setminus \bigcirc}$ | LTL |
| equivalence checking | PTIME <br> $\mathcal{O}(m \cdot \log \lvert S \rvert)$ | PTIME <br> $\mathcal{O}(m \cdot \log \lvert S \rvert)$ | PSPACE-complete |
| graph minimization | $\surd$ | $\surd$ | — |

where $AP$ is fixed and

$m$ = number of edges $\geq$ number of states = $|S|$