Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

Equivalences and Abstraction

- bisimulation
- CTL, CTL*-equivalence
- computing the bisimulation quotient $\longleftarrow$
- abstraction stutter steps
- simulation relations

Bisimulation quotient $\mathcal{T}/\sim$

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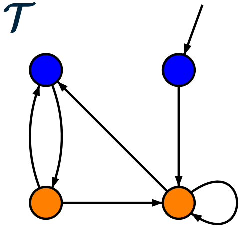

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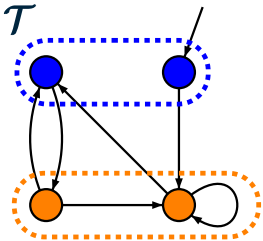

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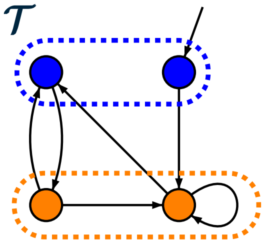

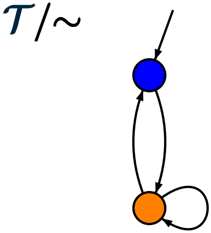

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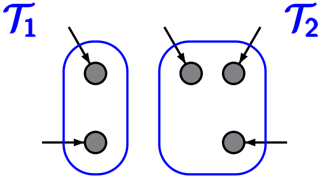

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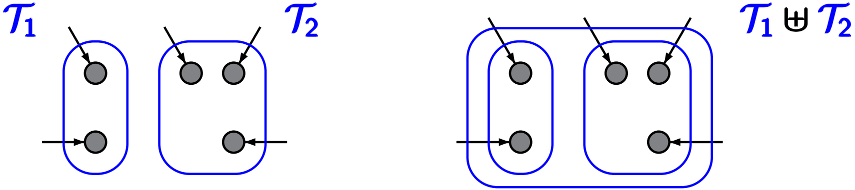

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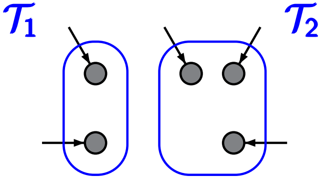

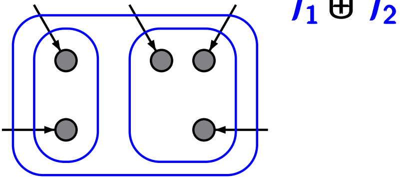

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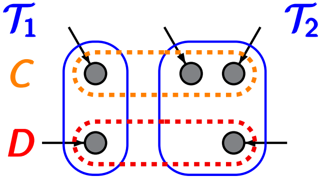

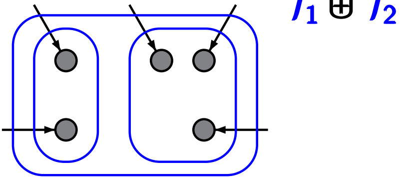

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2. graph minimization:

   replace $\mathcal{T}$ with $\mathcal{T}/\!\sim$ and analyze $\mathcal{T}/\!\sim$

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here: only explanations for finite transition systems, possibly with terminal states

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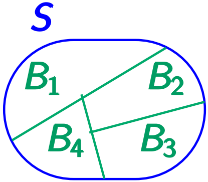

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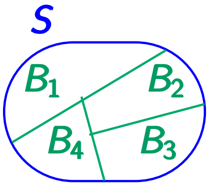

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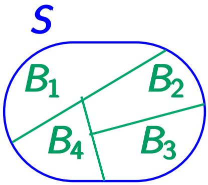

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A superblock denotes any union of blocks.

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- equivalence $\mathcal{R}$ on $S \rightsquigarrow$ partition $\mathcal{B} = S/\mathcal{R}$

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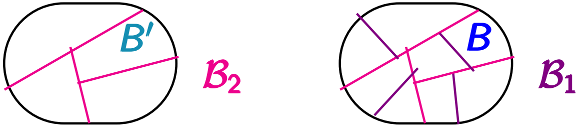

Example: if $\mathcal{R}$ is a bisimulation for $\mathcal{T}$ and an equivalence then $S/\mathcal{R}$ is *finer* than $S/\sim$

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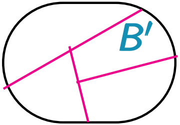
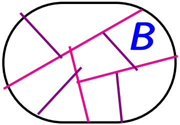


$\mathcal{B}_2$



$\mathcal{B}_1$

$\mathcal{B}_1$ is called *strictly finer* than $\mathcal{B}_2$ if

(1) $\mathcal{B}_1$ is finer than $\mathcal{B}_2$ and (2) $\mathcal{B}_1 \neq \mathcal{B}_2$

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by stepwise refinement of partitions the state set S

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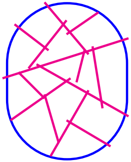

$S/\sim$

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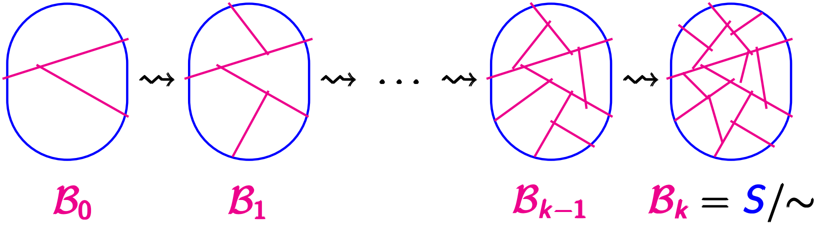

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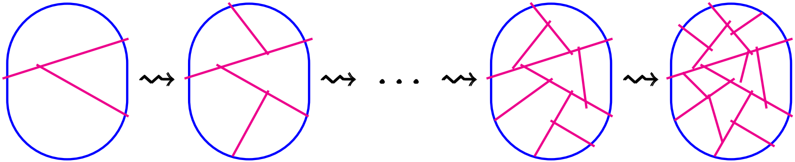

$\mathcal{B}_{AP} = \mathcal{B}_0$ $\quad \mathcal{B}_1 \quad \mathcal{B}_{k-1} \quad \mathcal{B}_k = S/\sim$

initial partition: $\mathcal{B}_{AP} = \mathcal{B}_0$

identifies all states with the same labeling

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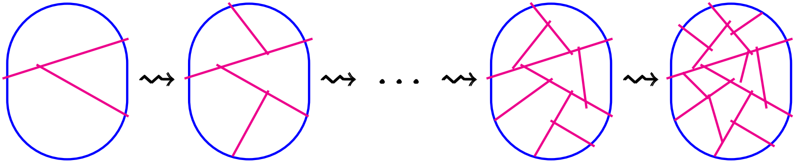

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initial partition: $\mathcal{B}_{AP} = \mathcal{B}_0 = S/\mathcal{R}_{AP}$ where

$$\mathcal{R}_{AP} = \{ (s_1, s_2) : L(s_1) = L(s_2) \}$$

Characterization of $S/\!\sim_{\mathcal{T}}$

... as the coarsest partition of the
state space S such that ....

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where $Pre(C) = \{s \in S : \exists s' \in C \text{ s.t. } s \to s'\}$

Partitioning refinement algorithm

input: finite TS $\mathcal{T}$ with state space S over AP (possibly with terminal states)

output: bisimulation quotient $S/\sim_{\mathcal{T}}$

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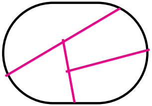
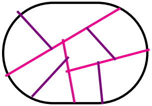
$\mathcal{B}_i$ 

$\mathcal{B}_{i+1}$ 

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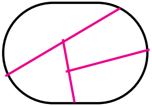
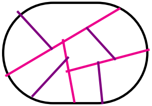
$\mathcal{B}_i$ 

$\mathcal{B}_{i+1}$ 

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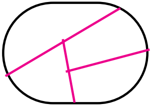
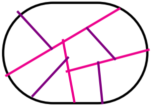
$\mathcal{B}_i$ 

$\mathcal{B}_{i+1}$ 

loop invariant:

$\mathcal{B}_i$ is coarser than $S/\!\sim_{\mathcal{T}}$ and finer than $\mathcal{B}_{AP}$

Maximal number of iterations?

$\mathcal{B}_0 := \mathcal{B}_{AP}$; $i := 0$

REPEAT

$\mathcal{B}_{i+1} := Refine(\mathcal{B}_i)$; $i := i+1$

UNTIL no further refinement is possible

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Assuming that $\mathcal{B}_i$ is strictly coarser than $\mathcal{B}_{i+1}$ for all i, what is the maximal number of refinement steps **?**

Maximal number of iterations?

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Assuming that $\mathcal{B}_i$ is strictly coarser than $\mathcal{B}_{i+1}$ for all i, what is the maximal number of refinement steps ?

answer: $|S| - 1$

Note that $|\mathcal{B}_i| \geq i+1$.

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answer: $|S| - 1$

Note that $|\mathcal{B}_i| \geq i+1$.

Hence: if there are $k = |S|-1$ iterations then $\mathcal{B}_k$ consists of singletons

initial partition $\mathcal{B}_{AP}$:

identifies all states s, t
s.t. $L(s) = L(t)$

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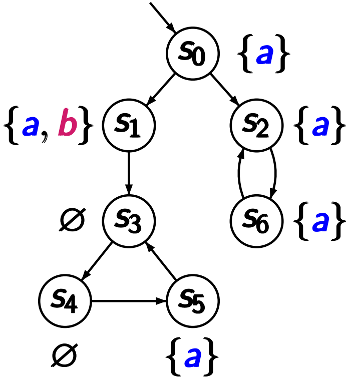

The initial partition

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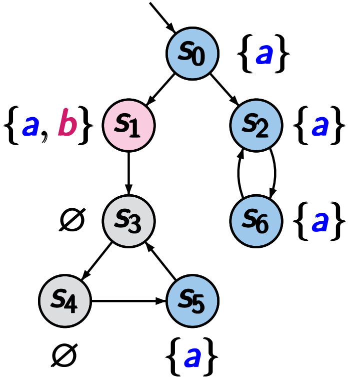

$$\mathcal{B}_{AP} = \Big\{ \{s_0, s_2, s_6, s_5\}, \{s_1\}, \{s_3, s_4\} \Big\}$$

initial partition $\mathcal{B}_{AP}$:

- identifies all states with the same labeling
- agrees with the quotient under the equivalence

$$s \equiv_{AP} t \quad \text{iff} \quad L(s) = L(t)$$

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> compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for AP

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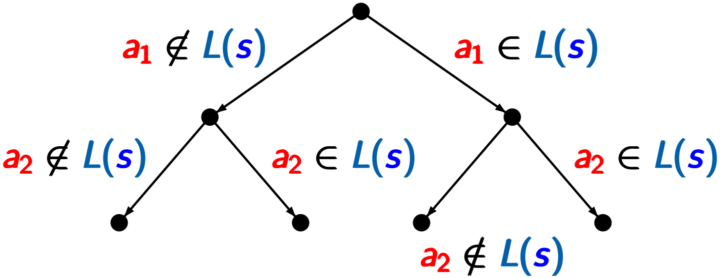

Computing the initial partition

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Computing the initial partition

compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, ..., a_k\}$

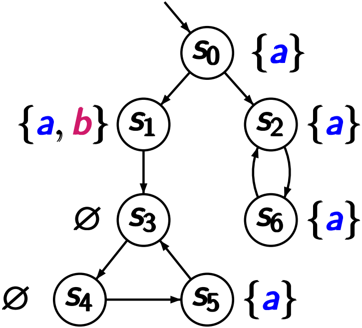
initally: each leaf represents the empty state-set

for each state s:

  traverse the decision tree from the root to a leaf v

  insert s in the set for v

Example: initial partition



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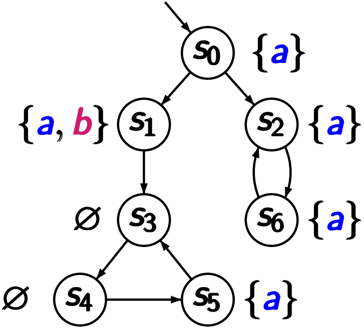

decision tree for
$AP = \{a, b\}$

1. level: $a \in L(s)$?
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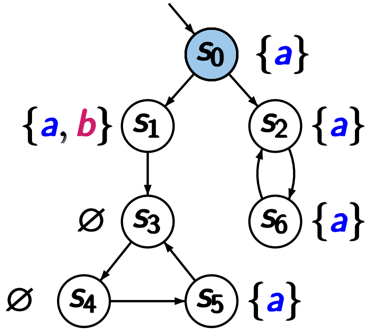
Example: initial partition



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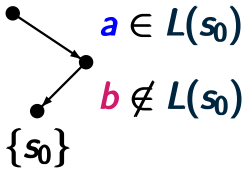

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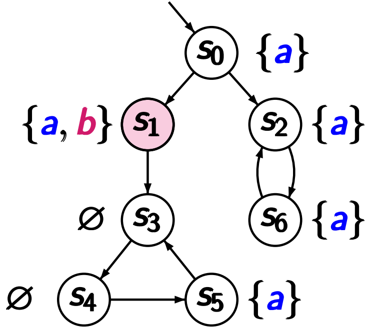

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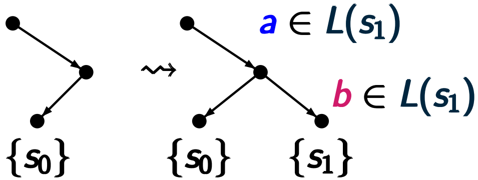

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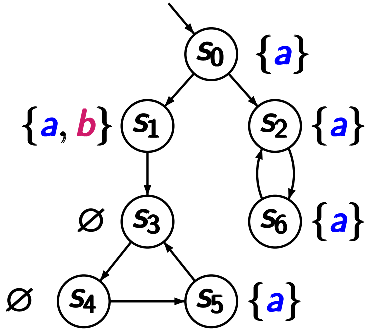

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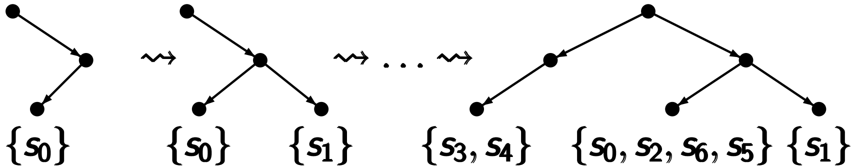

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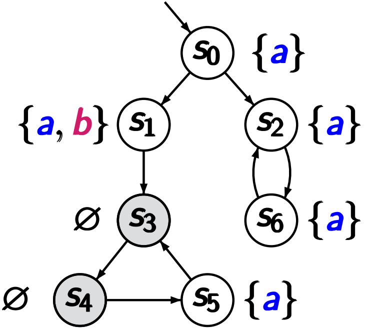

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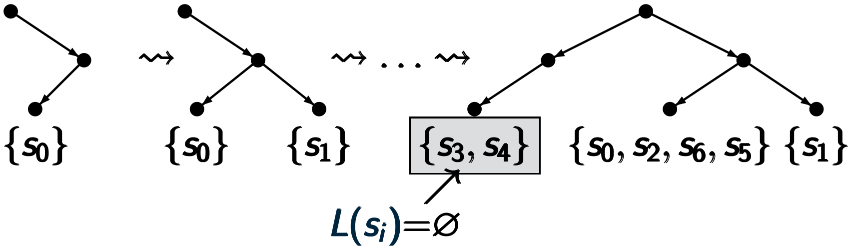

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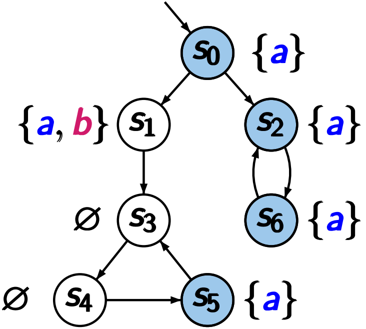

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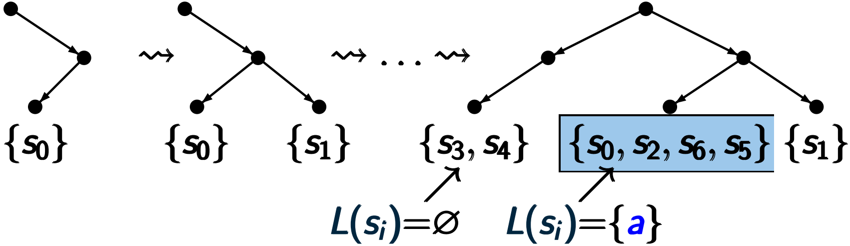

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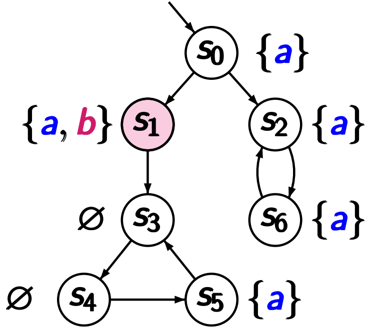

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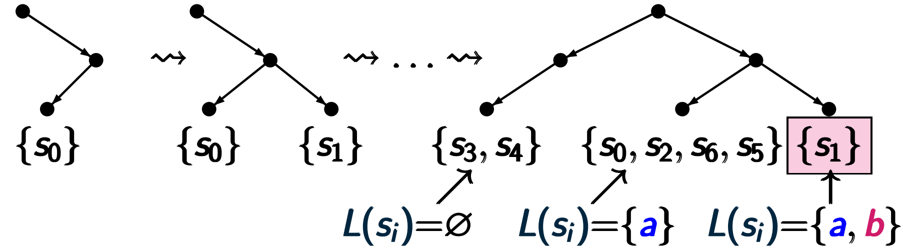

generate the root node v_0 of the decision tree

```
FOR ALL states s DO
   v := v0
   FOR i = 1, ..., k OD
      IF a_i ∈ L(s)
         THEN   v := find_or_add(right son of v)
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      FI
   OD

OD
```

suppose

$AP = \{a_1, \ldots, a_k\}$

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The state-sets of the leaves are the blocks in $\mathcal{B}_{AP}$.

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complexity: $\mathcal{O}(|S| \cdot |AP|)$

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Partitioning refinement (schema)

$\mathcal{B} := \mathcal{B}_{AP}$

`WHILE` refinements are possible `DO`

$\qquad \mathcal{B} := \mathit{Refine}(\mathcal{B})$

`OD`

return $\mathcal{B}$

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refinement: stabilization for some superblock C of $\mathcal{B}$:

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split each block $B \in \mathcal{B}$ into two blocks:

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$\mathcal{B} := \mathcal{B}_{AP}$

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WHILE refinements are possible DO
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```
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Partitioning splitter algorithm


```
𝓑 := 𝓑_AP
WHILE refinements are possible DO
      choose some superblock C of 𝓑;
      𝓑 := Refine(𝓑, C)
OD
return 𝓑
```

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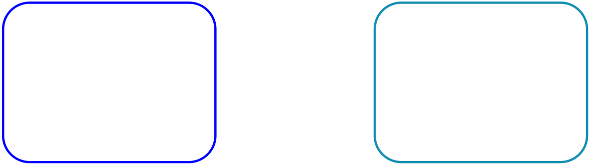
$Refine(B, C)$



block B superblock C

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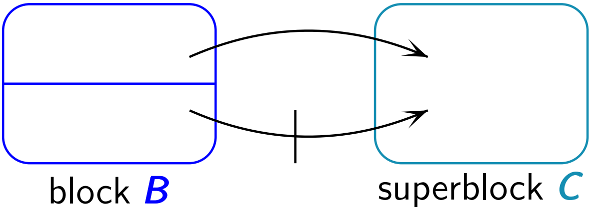

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$$\mathcal{B} := Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

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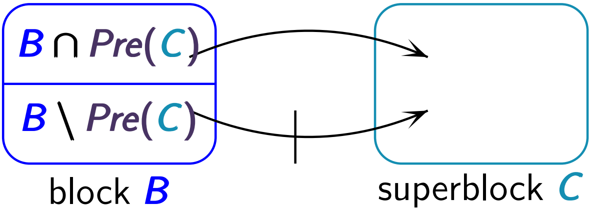
$$Refine(B, C) = \{B \cap Pre(C), B \setminus Pre(C)\}$$



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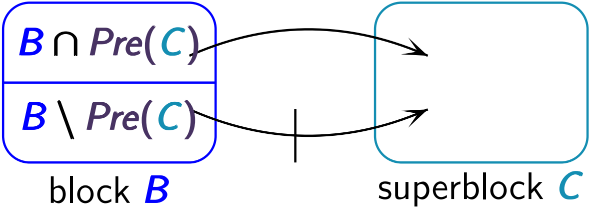

The refinement operator

Let $\mathcal{B}$ be a partition for S and C a superblock of $\mathcal{B}$.

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

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If $\mathcal{B}$ is finer than $\mathcal{B}_{AP}$ and coarser than $S/\sim_{\mathcal{T}}$ then:

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(a) $Refine(\mathcal{B}, C)$ is finer than $\mathcal{B}$

Let $\mathcal{B}$ be a partition for S and C a superblock of $\mathcal{B}$.

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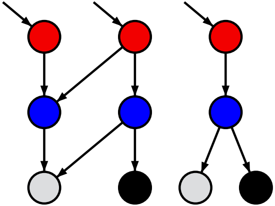
(b) $Refine(\mathcal{B}, C)$ is coarser than $S/{\sim_{\mathcal{T}}}$

(c) $Refine(\mathcal{B}, C) = \mathcal{B}$ for all $C \in \mathcal{B}$ iff $\mathcal{B} = S/{\sim_{\mathcal{T}}}$

Example: partitioning splitter algorithm



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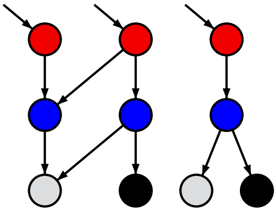

$\Longrightarrow$

refinement
w.r.t. ●

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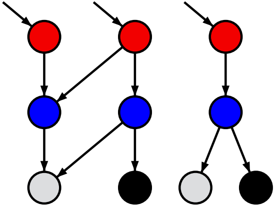

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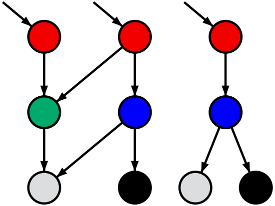

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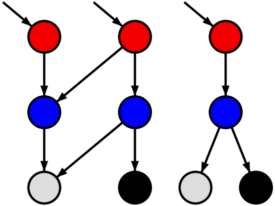

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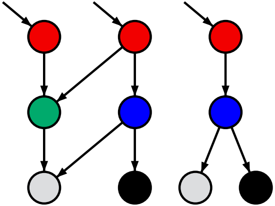

$\Downarrow$ refinement
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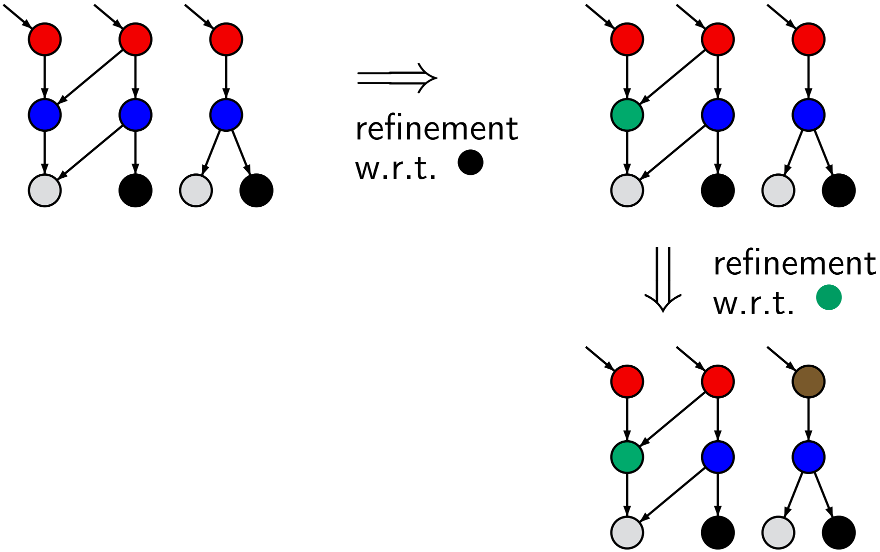

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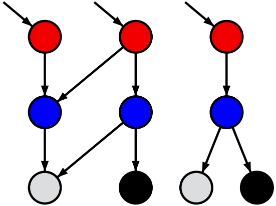
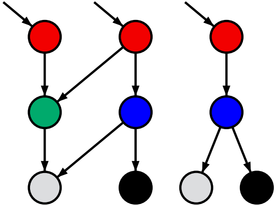


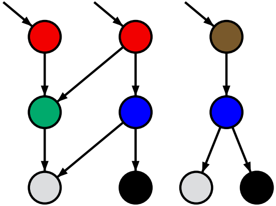
$\Longrightarrow$

refinement
w.r.t.



$\Downarrow$

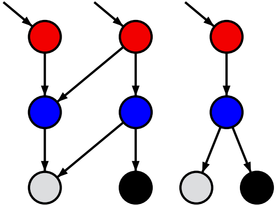
refinement
w.r.t.



$\Longleftarrow$

refinement
w.r.t.

Example: partitioning splitter algorithm



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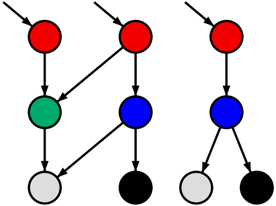

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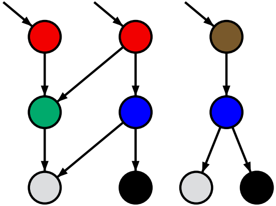

$\Longleftarrow$

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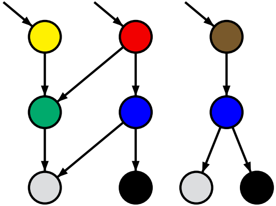
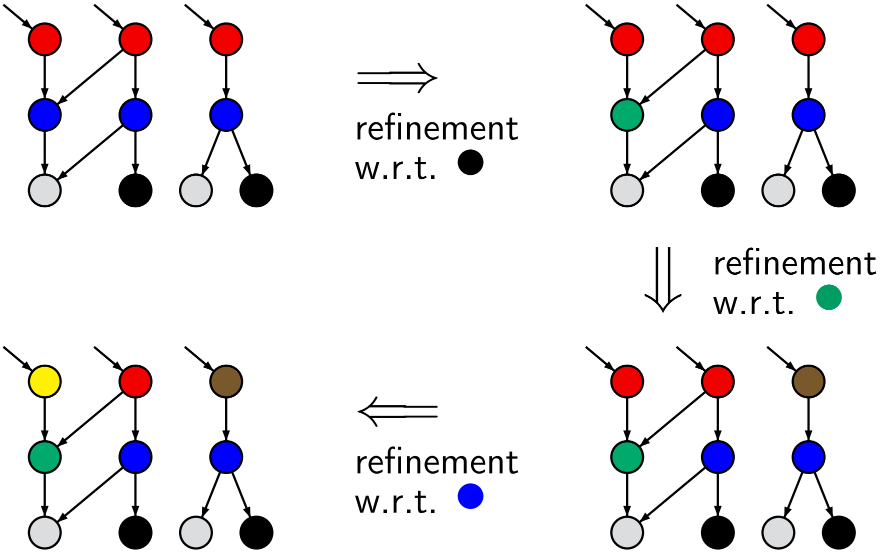


Example: partitioning splitter algorithm



7 bisimulation equivalence classes

The refinement operator

given a partition $\mathcal{B}$ and a superblock C of $\mathcal{B}$,
how to compute

$$\mathit{Refine}(\mathcal{B}, C)$$

efficiently **?**

The refinement operator

given a partition $\mathcal{B}$ and a superblock C of $\mathcal{B}$,
how to compute

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

efficiently **?**

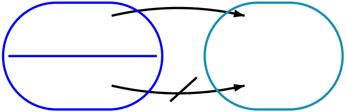
given a partition $\mathcal{B}$ and a superblock C of $\mathcal{B}$, how to compute

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

efficiently **?**

where for all blocks $B \in \mathcal{B}$:

$$Refine(B, C) = \{ B \cap Pre(C), B \setminus Pre(C) \} \setminus \{\varnothing\}$$



block B superblock C

Refinement operator $\mathit{Refine}(\mathcal{B}, C)$

Refinement operator $Refine(\mathcal{B}, C)$


```
FOR ALL s' ∈ C DO



OD
```

```
FOR ALL s' ∈ C DO
  FOR ALL s ∈ Pre(s') DO



  OD
OD
```

Refinement operator $Refine(\mathcal{B}, C)$

```
FOR ALL s' ∈ C DO
   FOR ALL s ∈ Pre(s') DO
         “move” state s from block [s]_B = B
                to the new block B ∩ Pre(C)
   OD
OD
```

Refinement operator $Refine(\mathcal{B}, C)$

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... states left in block $B \in \mathcal{B}$ belong to the new block $B \setminus Pre(C)$

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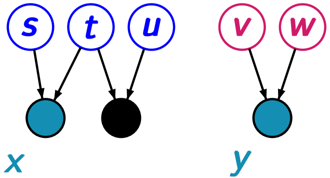
... states left in block $B \in \mathcal{B}$ belong to the new block $B \setminus Pre(C)$

time complexity:

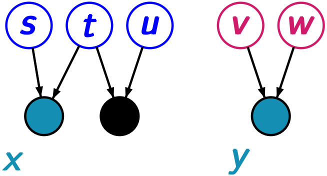
$$\mathcal{O}\Big(\sum_{s' \in C} |Pre(s')| + |C| \Big)$$

Example: refinement operator



partition $\mathcal{B}$

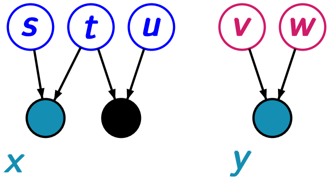
Example: refinement operator



superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow \mathit{Refine}(\mathcal{B}, C)$

Example: refinement operator



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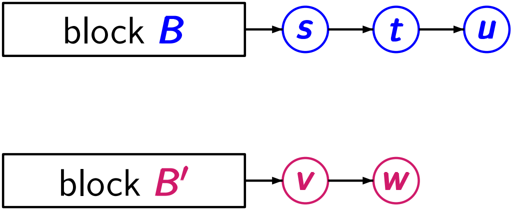
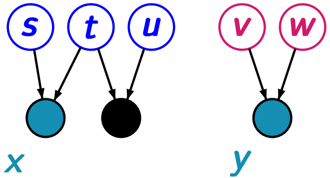


...

Example: refinement operator



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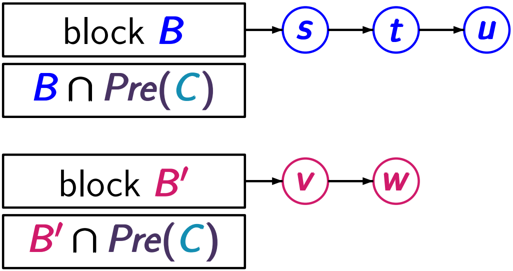

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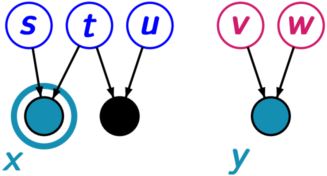

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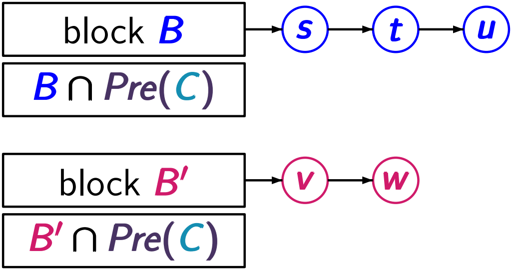
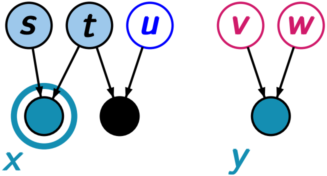


$\ldots$

Example: refinement operator



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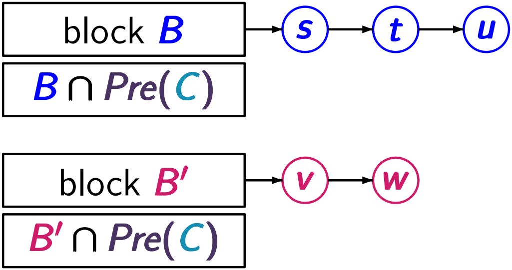

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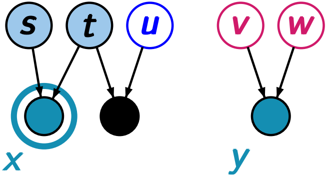

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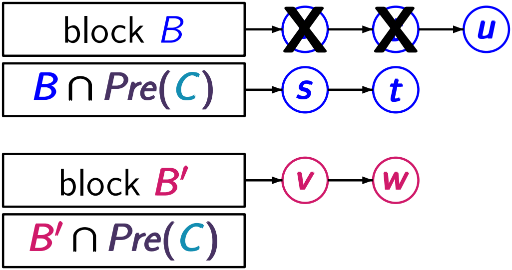

...

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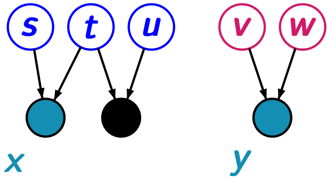

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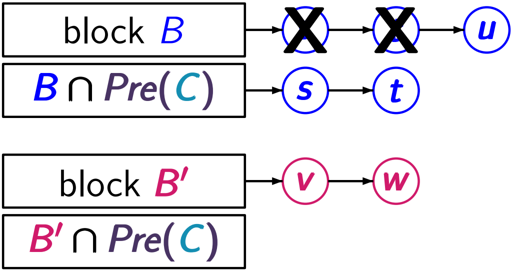
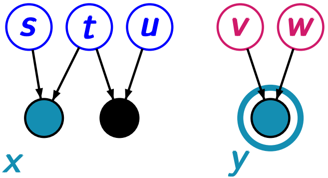


$\leftarrow$ new block $B \setminus Pre(C)$

...

Example: refinement operator



superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow \mathit{Refine}(\mathcal{B}, C)$

block B → ~~s~~ → ~~t~~ → u ← new block $B \setminus \mathit{Pre}(C)$

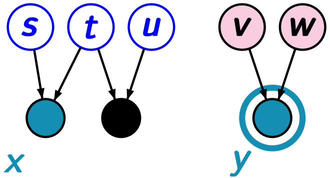
$B \cap \mathit{Pre}(C)$ → s → t

block B' → v → w

$B' \cap \mathit{Pre}(C)$

. . .

Example: refinement operator



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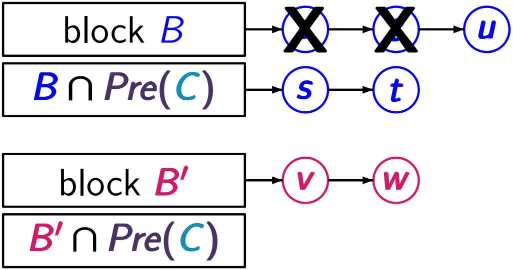

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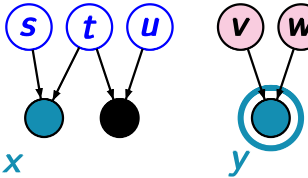

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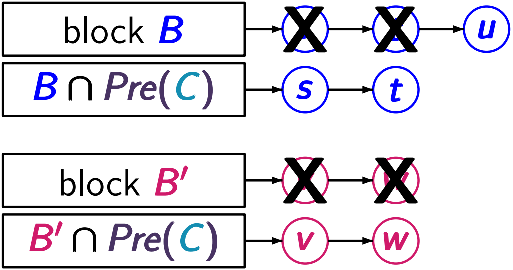

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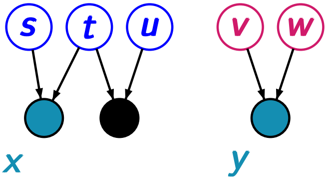

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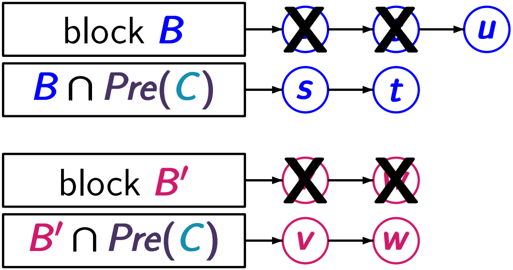
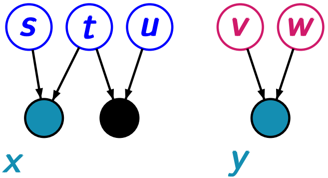


$\leftarrow$ new block $B \setminus \mathit{Pre}(C)$

$\leftarrow B' \setminus \mathit{Pre}(C) = \varnothing$

...

Example: refinement operator



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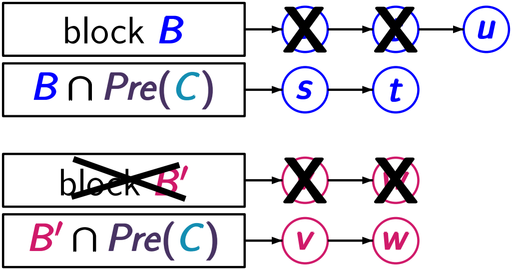

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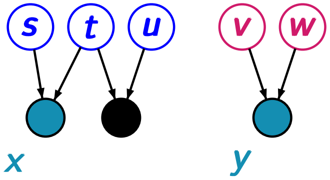

superblock $C = \{x, y\}$

$Refine(\mathcal{B}, C)$

$\boxed{B \setminus Pre(C)} \to u$

$\boxed{B \cap Pre(C)} \to s \to t$

$\boxed{B' \cap Pre(C)} \to v \to w$

$\ldots$

Partitioning splitter algorithm

```
B := B_AP

WHILE there is a splitter C for B DO

      select such a splitter C;
      B := Refine(B, C)
OD
return B
```

$\mathcal{B} := \mathcal{B}_{AP}$

WHILE there is a splitter C for $\mathcal{B}$ DO

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Partitioning splitter algorithm


```
𝓑 := 𝓑_AP          ⟵ time complexity: 𝒪(|S|·|AP|)
WHILE  there is a splitter C for 𝓑 DO
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OD
return 𝓑
```

Partitioning splitter algorithm

$\mathcal{B} := \mathcal{B}_{AP}$ $\longleftarrow$ $\boxed{\text{time complexity: } \mathcal{O}(|S| \cdot |AP|)}$

`WHILE` there is a splitter C for $\mathcal{B}$ `DO`

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`OD`

return $\mathcal{B}$

each state $s' \in C$ causes the costs $\mathcal{O}(|Pre(s')| + 1)$

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time complexity:

$$\mathcal{O}\Big(\sum_{C}\big(\sum_{s'\in C}|Pre(s')| + |C|\big) \ + \ |S|\cdot|AP|\Big)$$

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$+$ cost for splitter search and management

2 instances of the partitioning splitter algorithm
that differ in the choice and management of splitters

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- *Kanellakis-Smolka algorithm:*

  refinement according to all blocks of the partition of the previous iteration

Partitioning splitter algorithms

2 instances of the partitioning splitter algorithm that differ in the choice and management of splitters

- *Kanellakis-Smolka algorithm:*

  refinement according to all blocks of the partition of the previous iteration

- *Paige-Tarjan-algorithm:*

  simultaneous refinement according to 2 superblocks

Kanellakis-Smolka algorithm

Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$

REPEAT

  $\mathcal{B}_{\mathsf{old}} := \mathcal{B}$;

  FOR ALL $C \in \mathcal{B}_{\mathsf{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\mathsf{old}}$

return $\mathcal{B}$

Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

```
REPEAT
    B_old := B;
    FOR ALL  C ∈ B_old DO  B := Refine(B, C) OD
UNTIL  B = B_old
return B
```

Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

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    B_old := B;
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```

- maximal $|S|$ iterations

Kanellakis-Smolka algorithm


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𝓑 := 𝓑_AP;  𝓑_old := {S}      ⟵   cost: O(|S|·|AP|)
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- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs

$$\mathcal{O}(|Pre(s')| + 1)$$

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

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- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|Pre(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|)$

Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\text{old}} := \{S\}$ ⟵ cost: $\mathcal{O}(|S| \cdot |AP|)$

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REPEAT
    B_old := B;
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  if m = number of edges $= \sum_{s'} |Pre(s')|$

Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{old} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

REPEAT

  $\mathcal{B}_{old} := \mathcal{B}$;

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UNTIL $\mathcal{B} = \mathcal{B}_{old}$

return $\mathcal{B}$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|Pre(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|) = \mathcal{O}(m)$

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Kanellakis-Smolka algorithm

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B := B_AP;  B_old := {S}
REPEAT
    B_old := B;
    FOR ALL  C ∈ B_old DO  B := Refine(B, C) OD
UNTIL  B = B_old
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```

time complexity:
$\mathcal{O}(|S| \cdot m + |S| \cdot |AP|)$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|Pre(s')| + 1)$$
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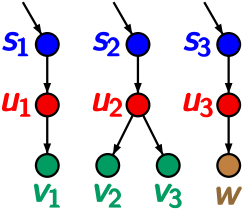
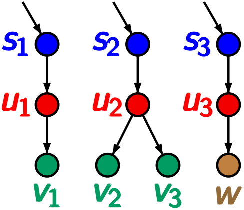
Example: Kanellakis-Smolka algorithm

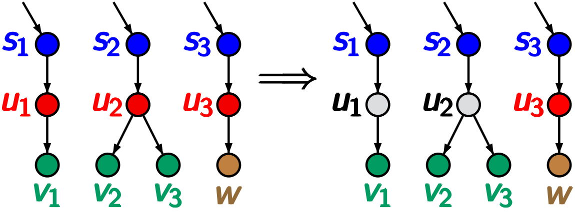




1. iteration:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$

Example: Kanellakis-Smolka algorithm



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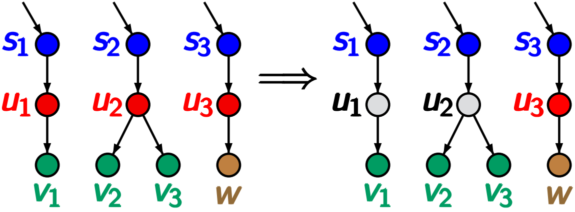

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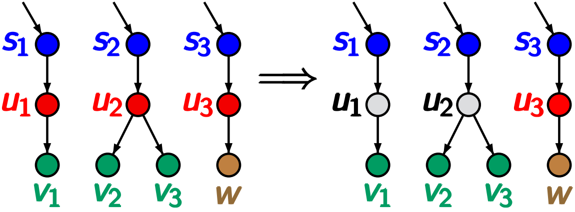
1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes

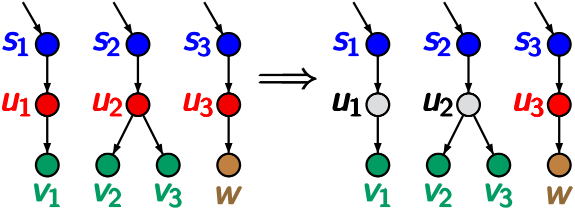
Example: Kanellakis-Smolka algorithm



1. iteration:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes
3. refinement w.r.t. $\{s_1, s_2, s_3\}$: no changes

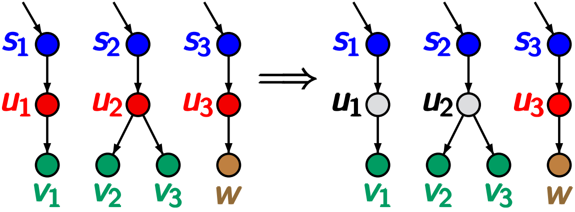
Example: Kanellakis-Smolka algorithm



1. iteration:

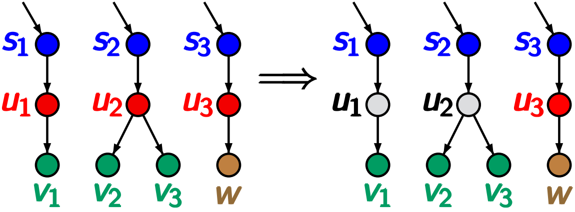
1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes
3. refinement w.r.t. $\{s_1, s_2, s_3\}$: no changes
4. refinement w.r.t. $\{u_1, u_2, u_3\}$: no changes

Example: Kanellakis-Smolka algorithm



2. iteration:

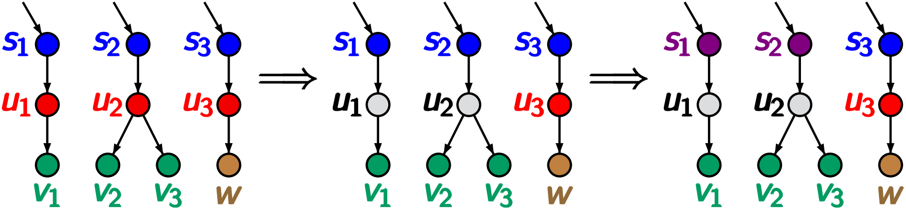
Example: Kanellakis-Smolka algorithm



2. iteration:

1. refinement w.r.t. $\{u_3\}$

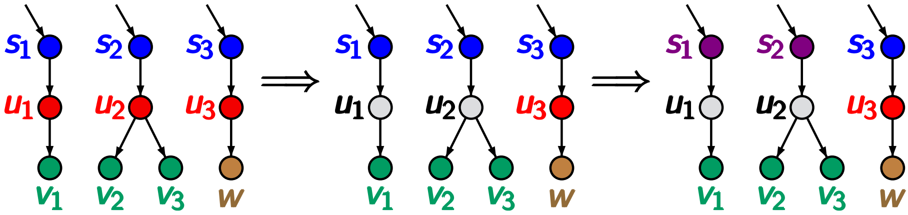
Example: Kanellakis-Smolka algorithm



2. iteration:

1. refinement w.r.t. $\{u_3\}$

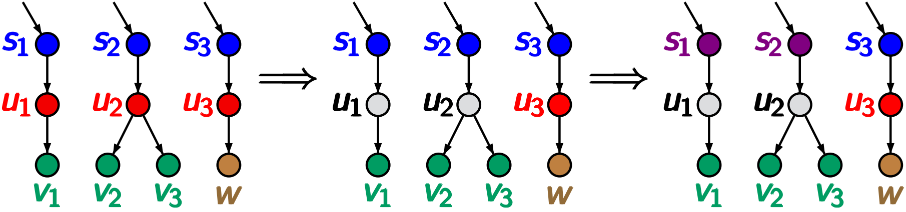
Example: Kanellakis-Smolka algorithm



2. iteration:

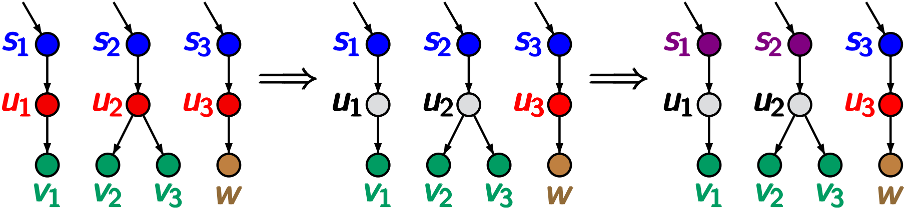
1. refinement w.r.t. $\{u_3\}$
2. refinement w.r.t. other blocks of the first iteration: no changes

Example: Kanellakis-Smolka algorithm



3. iteration:

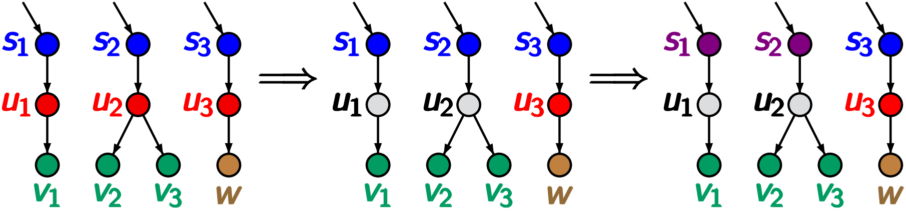
Example: Kanellakis-Smolka algorithm



3. iteration:

refinement w.r.t. all blocks of the second iteration:
no changes

Example: Kanellakis-Smolka algorithm



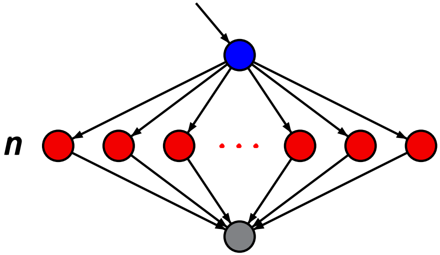
3. iteration:

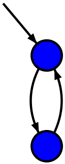
refinement w.r.t. all blocks of the second iteration: no changes

6 bisimulation equivalence classes:

$\{s_1, s_2\}, \{s_3\}, \{u_1, u_2\}, \{u_3\}, \{v_1, v_2, v_3\}, \{w\}$

Partitioning splitter algorithm





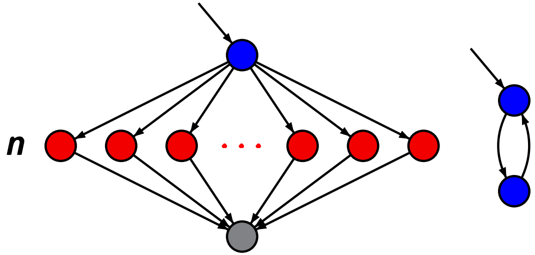
$AP = \{a, b\}$

● $\widehat{=} \{a\}$

● $\widehat{=} \{b\}$

● $\widehat{=} \varnothing$

Partitioning splitter algorithm



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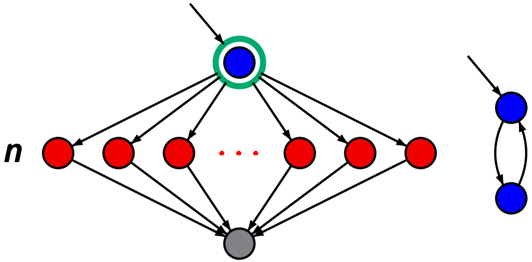
refinement w.r.t. ●:



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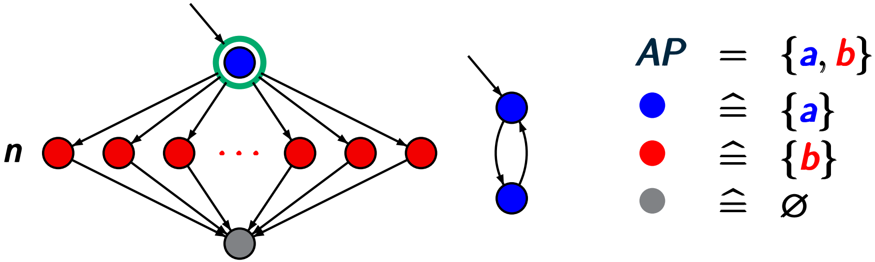
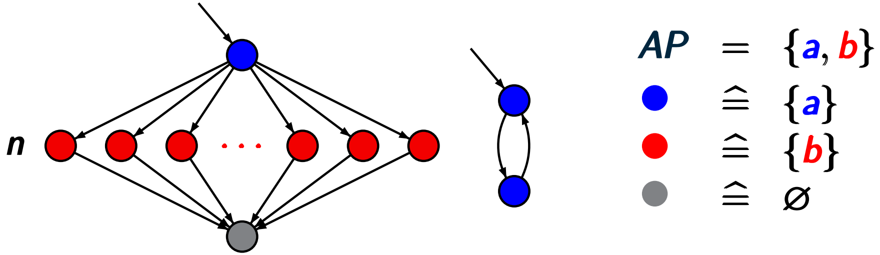
Partitioning splitter algorithm



refinement w.r.t. ●: causes the costs

$$\sum_{s'} \left| Pre(s') \right| = n$$



refinement w.r.t. ●: causes the costs

$$\sum_{s'} \left| Pre(s') \right| = n$$

alternatively: refinement w.r.t. ●: constant costs

Partitioning splitter algorithms

Kanellakis-Smolka algorithm:

initially: $\mathcal{B}_{\mathsf{old}} = \mathcal{B} = \mathcal{B}_{AP}$

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loop invariant:

(1) $\mathcal{B}$ finer than $\mathcal{B}_{\mathsf{old}}$ and coarser than $S/\sim$

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initially: $\mathcal{B}_{\mathsf{old}} = \{S\}$

Partitioning splitter algorithms

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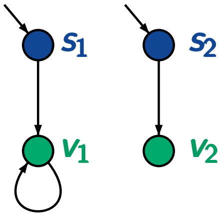
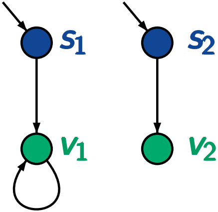
iteration: ternary refinement operator

initially: $\mathcal{B}_{\mathsf{old}} = \{S\}$, $\mathcal{B} = \mathit{Refine}(\mathcal{B}_{AP}, S)$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



s_1
s_2
v_1
v_2



state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

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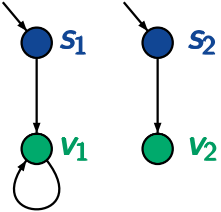

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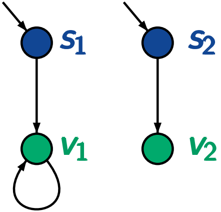
$Pre(S)$ = set of nonterminal states

$= \{s_1, s_2, v_1\}$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



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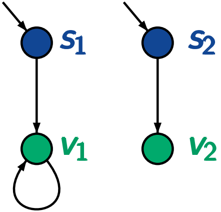
$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

$Pre(S)$ = set of nonterminal states $= \{s_1, s_2, v_1\}$

$$\{v_1, v_2\} \cap Pre(S) = \{v_1\}$$
$$\{v_1, v_2\} \setminus Pre(S) = \{v_2\}$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



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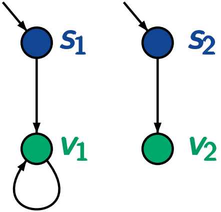
initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S)$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



state space $S = \{s_1, s_2, v_1, v_2\}$

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initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S) = \Big\{ \{s_1, s_2\}, \{v_1\}, \{v_2\} \Big\}$$

Paige-Tarjan algorithm

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

```
WHILE 𝓑 ≠ 𝓑_old DO



OD
```

Paige-Tarjan algorithm

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

   select a block $C' \in \mathcal{B}_{\mathsf{old}} \setminus \mathcal{B}$;

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

`WHILE` $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ `DO`

  select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

  select a block $C \in \mathcal{B}$ with $C \subseteq C'$

`OD`

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

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OD

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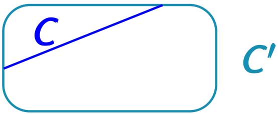

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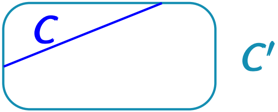


WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

  select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

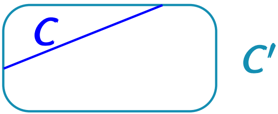
  select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

refine $\mathcal{B}$
w.r.t. C and $C' \setminus C$

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \textit{Refine}(\mathcal{B}_{AP}, S)$;




```
WHILE B ≠ B_old DO
```

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

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$\mathcal{B} := \textit{Refine}(\mathcal{B}, C)$

$\mathcal{B} := \textit{Refine}(\mathcal{B}, C')$

refine $\mathcal{B}$
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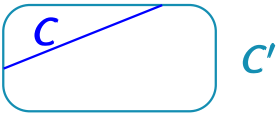
```
OD
```

Paige-Tarjan algorithm

$\mathcal{B}_{old} := \{S\}$; $\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$;



WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select a block $C' \in \mathcal{B}_{old} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := Refine(\mathcal{B}, C)$

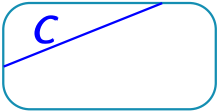
$\mathcal{B} := Refine(\mathcal{B}, C')$

refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;



C'

WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

OD

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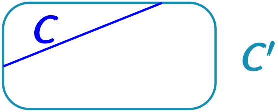

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$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

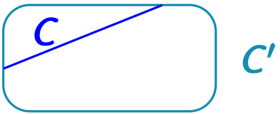
refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \textit{Refine}(\mathcal{B}_{AP}, S)$;



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  select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

  select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

  $\mathcal{B} := \textit{Refine}(\mathcal{B}, C, C' \setminus C)$

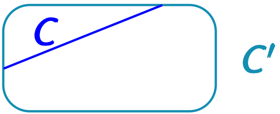
> refine $\mathcal{B}$ simultaneously
> w.r.t. C and $C' \setminus C$

  add C and $C' \setminus C$ to $\mathcal{B}_{\text{old}}$ and remove C' from $\mathcal{B}_{\text{old}}$

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;



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$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

add C and $C' \setminus C$ to $\mathcal{B}_{\text{old}}$ and remove C' from $\mathcal{B}_{\text{old}}$

OD

loop invariant: $\mathcal{B}$ is stable w.r.t. each block in $\mathcal{B}_{\text{old}}$

The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'

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The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
- C a block in $\mathcal{B}$ s.t. $C \subseteq C'$

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
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simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

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simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
- C a block in $\mathcal{B}$ s.t. $C \subseteq C'$

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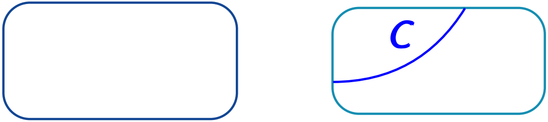
where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



block B

superblock C'

The ternary refinement operator

Let $\mathcal{B}$ be a partition and

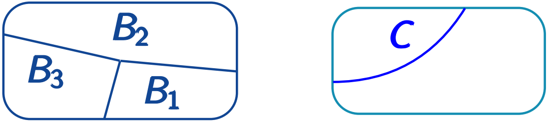
- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
- C a block in $\mathcal{B}$ s.t. $C \subseteq C'$

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



block B superblock C'

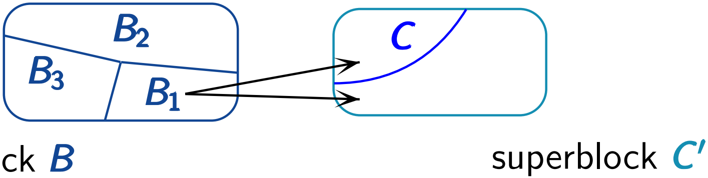
The ternary refinement operator

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$\mathit{Refine}(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} \mathit{Refine}(B, C, C' \setminus C)$$

where for block $B \subseteq \mathit{Pre}(C')$:

$$\mathit{Refine}(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



$$B_1 \;=\; B \cap \mathit{Pre}(C) \cap \mathit{Pre}(C' \setminus C)$$

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

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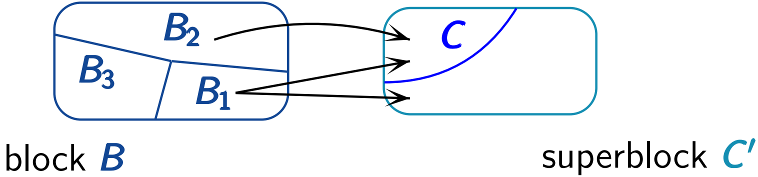
$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



$$\begin{aligned} B_1 &= B \cap Pre(C) \cap Pre(C' \setminus C) \\ B_2 &= (B \cap Pre(C)) \setminus Pre(C' \setminus C) \end{aligned}$$

The ternary refinement operator

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

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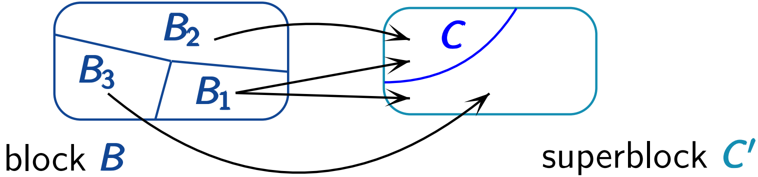


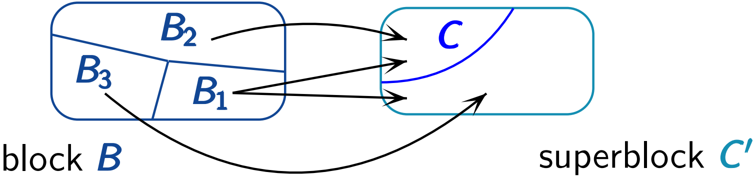
$$\begin{aligned} B_1 &= B \cap Pre(C) \cap Pre(C' \setminus C) \\ B_2 &= (B \cap Pre(C)) \setminus Pre(C' \setminus C) \\ B_3 &= (B \cap Pre(C' \setminus C)) \setminus Pre(C) \end{aligned}$$

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



for block B with $B \cap Pre(C') = \varnothing$:

$$Refine(B, C, C' \setminus C) = \{B\}$$

Stability of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

Suppose that for all blocks $B \in \mathcal{B}$:

$$B \subseteq \mathit{Pre}(C') \quad \text{or} \quad B \cap \mathit{Pre}(C') = \varnothing$$

Stability of $Refine(\mathcal{B}, C, C' \setminus C)$

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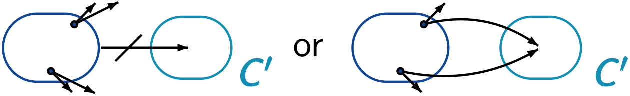

Stability of *Refine*$(\mathcal{B}, C, C' \setminus C)$

Suppose that for all blocks $B \in \mathcal{B}$:

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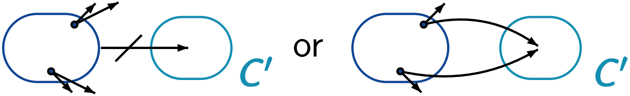
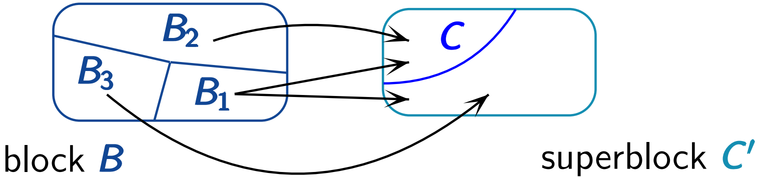
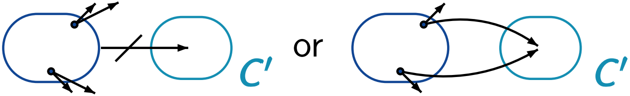


Then the new blocks B_1, B_2, B_3 in *Refine*$(B, C, C' \setminus C)$ are stable w.r.t. the superblocks C and $C' \setminus C$.



Suppose that for all blocks $B \in \mathcal{B}$:

$$B \subseteq Pre(C') \quad \text{or} \quad B \cap Pre(C') = \varnothing$$

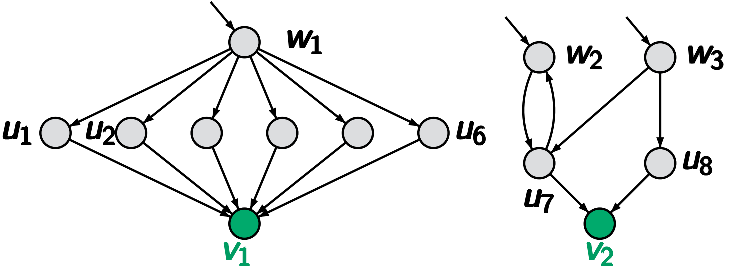


Then the new blocks B_1, B_2, B_3 in $Refine(B, C, C' \setminus C)$ are stable w.r.t. the superblocks C and $C' \setminus C$.

If $\mathcal{B}$ is stable w.r.t. all blocks in $\mathcal{B}_{old}$ and $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subsetneq C'$ then $Refine(\mathcal{B}, C, C' \setminus C)$ is stable w.r.t. all blocks in the partition

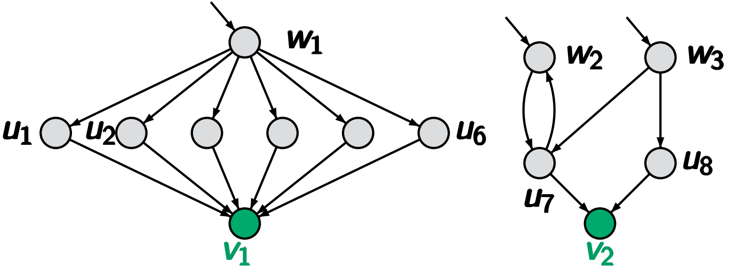
$$(\mathcal{B}_{old} \setminus \{C'\}) \cup \{C, C' \setminus C\}$$

Example: Paige-Tarjan algorithm



$$AP = \{green, gray\}, \quad \mathcal{B}_{old} = \{S\}$$

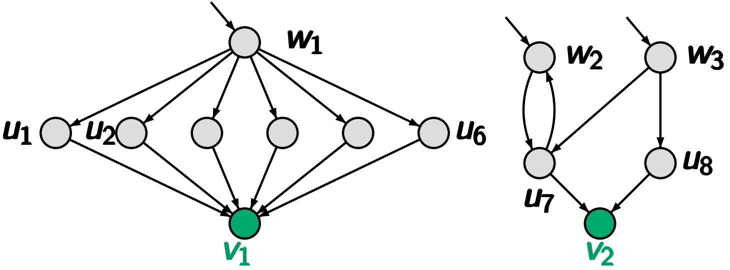
Example: Paige-Tarjan algorithm



$$AP = \{green, gray\}, \quad \mathcal{B}_{old} = \{S\}$$

initial partition:

$$\begin{aligned} \mathcal{B}_0 &= Refine(\mathcal{B}_{AP}, S) = \mathcal{B}_{AP} \\ &= \big\{\{v_1, v_2\}, \{u_1, \ldots, u_8, w_1, w_2, w_3\}\big\} \end{aligned}$$



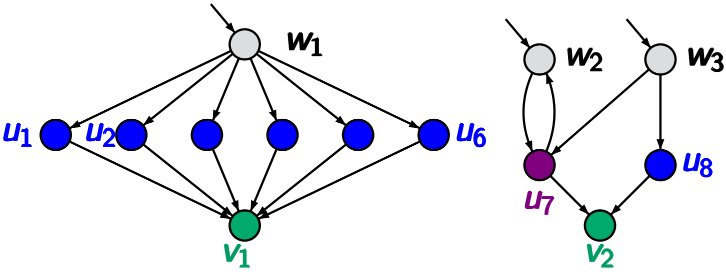
initially: $\mathcal{B}_{\text{old}} = \{S\}$

$\mathcal{B}_0 = \{\{v_1, v_2\}, \{u_1, \ldots, u_8, w_1, w_2, w_3\}\}$

first refinement step:

$Refine(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\})$

Example: Paige-Tarjan algorithm



initially: $\mathcal{B}_{\text{old}} = \{S\}$

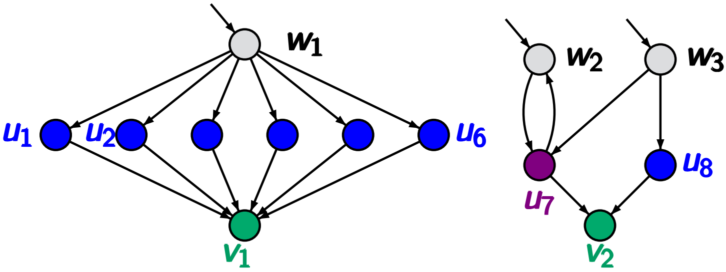
$$\mathcal{B}_0 = \Big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\Big\}$$

first refinement step:

$$\mathit{Refine}(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\}) =$$

$$\mathcal{B}_1 = \Big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\Big\}$$

Example: Paige-Tarjan algorithm



initially: $\mathcal{B}_{\text{old}} = \{S\}$

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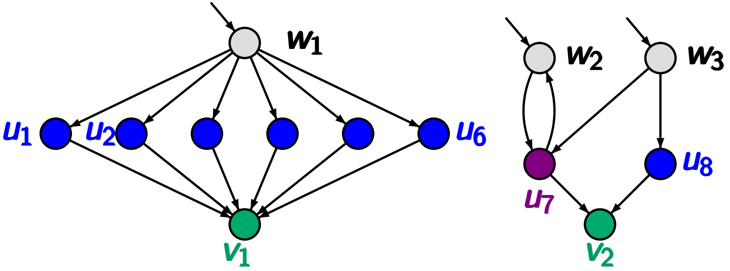
first refinement step:

$$\mathit{Refine}(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\}) =$$

$$\mathcal{B}_1 = \big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\big\}$$

$$\mathcal{B}_{\text{old}} = \big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\big\}$$

Example: Paige-Tarjan algorithm



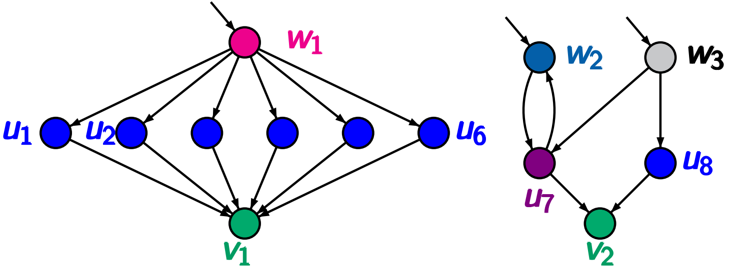
first refinement step:

$$\mathcal{B}_1 = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\}$$
$$\mathcal{B}_{\text{old}} = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\}$$

second refinement step:

$$\mathit{Refine}(\mathcal{B}_1, ?, ?)$$

Example: Paige-Tarjan algorithm



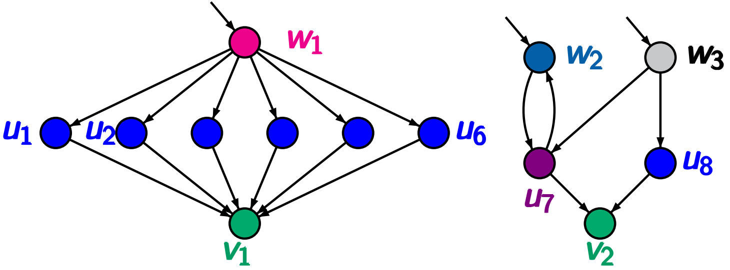
first refinement step:

$$\mathcal{B}_1 = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\}$$
$$\mathcal{B}_{old} = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\}$$

second refinement step:

$$\mathit{Refine}(\mathcal{B}_1, \{u_7\}, \{u_1, \ldots, u_6, u_8, w_1, w_2, w_3\})$$

Example: Paige-Tarjan algorithm



first refinement step:

$$\mathcal{B}_1 = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\}$$
$$\mathcal{B}_{\text{old}} = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\}$$

second refinement step:

$$\mathit{Refine}(\mathcal{B}_1, \{u_7\}, \{u_1, \ldots, u_6, u_8, w_1, w_2, w_3\})$$
$$= \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1\}, \{w_2\}, \{w_3\}\}$$

Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;
```
WHILE
```
$\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$
```
DO
```
select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$
```
OD
```
return $\mathcal{B}$

Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;
WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO
  select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;
  add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$
  $\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$
OD
return $\mathcal{B}$

efficient implementation of $\mathit{Refine}(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}\big(|C| + |\mathit{Pre}(C)|\big)$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

```
WHILE
```
$\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

  select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

  add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$

  $\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$

OD

return $\mathcal{B}$

efficient implementation of $\mathit{Refine}(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}\big(|C| + |\mathit{Pre}(C)|\big)$ uses counters

$$\delta(s, D) = |\mathit{Post}(s) \cap D| \text{ for } D \in \mathcal{B}_{\mathsf{old}}$$

Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$D \in \mathcal{B}_{\text{old}}$

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$s \in Pre(D)$ $\qquad$ $D \in \mathcal{B}_{\text{old}}$

Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$$s \in Pre(D) \qquad D \in \mathcal{B}_{\mathsf{old}}$$

step 1: compute $\delta(\ldots)$ for the new blocks C and $C' \setminus C$ in $\mathcal{B}_{\mathsf{old}}$

Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$s \in Pre(D)$ $\qquad$ $D \in \mathcal{B}_{\text{old}}$

step 1: compute $\delta(\ldots)$ for the new blocks C and $C' \setminus C$ in $\mathcal{B}_{\text{old}}$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of *Refine*$(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$

step 2: compute *Refine*$(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C) \leftarrow \boxed{\text{for } s \in Pre(C')}$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in \mathit{Pre}(C')}$

$$\begin{aligned} \delta(s, C) \quad &= |\mathit{Post}(s) \cap C| \\ \delta(s, C' \setminus C) &= |\mathit{Post}(s) \cap (C' \setminus C)| \end{aligned}$$

step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C), \delta(s, C' \setminus C) \leftarrow \boxed{\text{for } s \in \mathit{Pre}(C')}$

$$\delta(s, C) \qquad = |\mathit{Post}(s) \cap C|$$
$$\delta(s, C' \setminus C) = |\mathit{Post}(s) \cap (C' \setminus C)|$$

step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \cap \mathit{Pre}(C') = \varnothing$ we have:

$$\mathit{Refine}(B, C, C' \setminus C) = \{B\}$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\leftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) = |Post(s) \cap C|$$
$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\leftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\begin{aligned} \delta(s, C) &= |Post(s) \cap C| \\ \delta(s, C' \setminus C) &= |Post(s) \cap (C' \setminus C)| \end{aligned}$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

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step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$
$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$
$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) \quad = |Post(s) \cap C|$$

$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$

$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$

$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) \qquad = |Post(s) \cap C|$$

$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$

$$B_2 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) = 0\}$$

$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C) \leftarrow \boxed{\text{for } s \in \mathit{Pre}(C')}$

$$\delta(s, C) \qquad = |\mathit{Post}(s) \cap C|$$

$$\delta(s, C' \setminus C) = |\mathit{Post}(s) \cap (C' \setminus C)|$$

step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

> for $B \in \mathcal{B}$ with $B \subseteq \mathit{Pre}(C')$:
>
> $$\mathit{Refine}(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$

$$B_2 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) = 0\}$$

$$B_3 = \{s \in B : \delta(s, C) = 0, \delta(s, C' \setminus C) > 0\}$$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;

`WHILE` $\mathcal{B} \neq \mathcal{B}_{old}$ `DO`

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{old}$ and remove C' from $\mathcal{B}_{old}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

`OD`

Paige-Tarjan algorithm

$\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;

FOR ALL $s \in S$ DO $\delta(s, S) := |Post(s)|$ OD

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{old}$ and remove C' from $\mathcal{B}_{old}$

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    FOR ALL $s \in \mathit{Pre}(C)$ DO $\delta(s, C) := 0$ OD
    FOR ALL $s' \in C$ DO
      FOR ALL $s \in \mathit{Pre}(s')$ DO $\delta(s, C) := \delta(s, C)+1$ OD
    OD

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    OD
    FOR ALL $s \in Pre(C)$ DO
        $\delta(s, C' \backslash C) := \delta(s, C') - \delta(s, C)$ OD
  $\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)$
OD

Complexity of the Paige-Tarjan algorithm

let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a finite TS

n = # states $= |S|$

m = # transitions

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in what follows, we suppose $m \geq n$

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total cost for all refinement operations: $\mathcal{O}(m \cdot \log n)$