# Concurrency Theory

## Lecture 18: True Concurrency Semantics of Petri Nets (2)

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http://moves.rwth-aachen.de/teaching/ws-19-20/ct

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# Overview

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- Partial-order semantics of Petri nets = set of distributed runs
  - a distributed run is an acyclic (causal) net which contains no choices
  - a distributed run is a partial ordering of transition occurrences
- Today: the set of all distributed runs can be represented by a specific branching process, the unfolding

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- A net unfolding is the true concurrency counterpart of a marking graph.
- It is the unique maximal branching process in a complete lattice.
- The reachable markings of a 1-bounded net are covered by a finite prefix of this maximal branching process.

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# Overview

# Elementary system nets

## Net

An elementary net system $N$ is a tuple $(P, T, F, M_0)$ where:

- $P$ is a countable set of places
- $T$ is a countable set of transitions with $P \cap T = \varnothing$
- $F \subseteq (P \times T) \cup (T \times P)$ are the arcs satisfying:

$$\forall t \in T.\ {}^\bullet t \text{ and } t^\bullet \text{ are finite and non-empty}$$

- $M_0 : P \to \mathbb{N}$ is the initial marking.

Places and transitions are generically called nodes.

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Assumption: (possibly) infinite elementary nets are 1-bounded. Thus any marking can be viewed as a subset of places.

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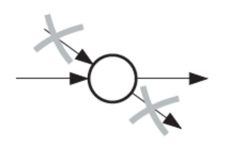
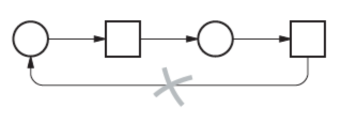
**Intuition**

No place branches, no sequence of arcs forms a loop, and each sequence of arcs has a first node.





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3. for each node $x \in Q \cup V$, the set $\{ y \mid (y, x) \in G^+ \}$ is finite
4. $M_0$ equals the minimal set of places in $K$ under $G^+$, i.e.,

$$M_0 \; = \; {}^\circ K \; = \; \{ q \in Q \mid {}^\bullet q = \varnothing \}.$$

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Every causal net is one-bounded, i.e., in every marking every place will hold at most one token.

## Absence of superfluous places and transitions

Every causal net has a step sequence that visits all places and fires every transition.

# What is a distributed run?

## Distributed run

A distributed run of a one-bounded elementary net system $N$ is:

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Examples on the black board.

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Examples on the black board.

Today: a characterization of distributed runs using homomorphisms.

# Net homomorphisms

[2] Here $h(X)$ for set $X$ of nodes is defined by $h(X) = \bigcup_{x \in X} h(x)$.
[3] Due to the 1-boundedness, a marking $M$ is a subset of the set $P$ of places.

# Net homomorphisms

## Homomorphism

A homomorphism from $N_1 = (P_1, T_1, F_1, M_{0,1})$ to $N_2 = (P_2, T_2, F_2, M_{0,2})$ is a mapping $h : P_1 \cup T_1 \to P_2 \cup T_2$ such that:

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## Intuition

A homomorphism is a mapping between nets that preserves the nature of nodes and the environment of nodes. A homomorphism from $N_1$ to $N_2$ means that $N_1$ can be folded onto a part of $N_2$, or in other words, that $N_1$ can be obtained by partially unfolding a part of $N_2$.

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[3] Due to the 1-boundedness, a marking $M$ is a subset of the set $P$ of places.

# Distributed run using homomorphisms

**Distributed run** **[Best and Fernandez, 1988]**

A distributed run of an elementary net system $N$ is a pair $(K, h)$ where $K$ is a causal net and $h$ is a homomorphism from $K$ to $N$.[4]

[4] Best and Fernandez called this a process of a net.

[5] In the previous lecture, the labeling $h$ was explicitly given as $\ell$.

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A distributed run $(K, h)$ of $N$ may be viewed as a net $K$ of which the places and transitions are labeled by places and transitions of $N$

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A distributed run $(K, h)$ of $N$ may be viewed as a net $K$ of which the places and transitions are labeled by places and transitions of $N$ such that the labeling $h$ forms a net homomorphism from $K$ to $N$.[5]

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# Examples

# Overview

# Conflicts

A distributed run is based on a causal net. A branching process on an occurrence net. Main difference: the presence of conflicts (aka: choices).

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## Conflict

Let $N = (P, T, F, M)$ be a net. Nodes $x_1$ and $x_2$ are in conflict, denoted $x_1 \# x_2$, if there exist distinct transitions $t_1, t_2 \in T$ such that $^\bullet t_1 \cap pre t_2 \neq \varnothing$ and $(t_1, x_1) \in F^*$ and $(t_2, x_2) \in F^*$.

Node $x$ is in self-conflict whenever $x \# x$.

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Note that in a causal net $\# = \varnothing$ as $^\bullet t_1 \cap {}^\bullet t_2 = \varnothing$ for any two distinct transitions $t_1$ and $t_2$.

# Occurrence net

## Occurrence net

A net $K = (Q, V, G, M)$ is an occurrence net iff:

1. for each $q \in Q$, $|{}^\bullet q| \leqslant 1$
2. the transitive closure $G^+$ of $G$ is irreflexive
3. for each node $x \in Q \cup V$ we have $\{ y \mid (y, x) \in G^+ \}$ is finite
4. no transition $v \in V$ is in self-conflict
5. $M_0 = {}^\circ K \ = \ \{ q \in Q \mid \ {}^\bullet q = \varnothing \}$.

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## Remark

Since $\# = \varnothing$ in a causal net, and each causal net fulfils the remaining conditions, every causal net is an occurrence net.

# Example

# Branching process

**Branching process** **[Engelfriet 1991]**

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$$\forall v, v' \in Q. \ ({}^\bullet v = {}^\bullet v' \text{ and } h(v) = h(v') \quad \text{implies} \quad v = v') \, .$$

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**Examples**

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# Examples

# Properties of branching processes

Let $K$ be a branching process of net $N$. Then:

1. A place of $K$ can get marked at most once, and an event (aka: transition) of $K$ can occur at most once in any step sequence of $K$
2. For $Q' \subseteq Q$: $K$ has some reachable marking $M$ with $Q' \subseteq M$ if and only if all places in $Q'$ are pairwise concurrent.

Nodes $x, y$ are concurrent if neither $(x, y) \in G^+$, nor $(y, x) \in G^+$ nor $x \# y$.

# Overview

# Relating branching processes

## Homomorphisms and isomorphisms between branching processes

Let $B_1 = (K_1, h_1)$ and $B_2 = (K_2, h_2)$ be two branching processes of net $N$. A homomorphism from $B_1$ to $B_2$ is a homomorphism $h$ from $K_1$ to $K_2$ such that $h_2 \circ h = h_1$.[6]

[6] The composition of two net homomorphisms is a net homomorphism.

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An isomorphism is a bijective homomorphism. $B_1$ and $B_2$ are isomorphic if there is an isomorphism from $B_1$ to $B_2$.

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Being isomorphic is an equivalence relation. Its equivalence classes are called isomorphism classes.

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# Approximation of branching processes

## Approximation

Let $B_1$ and $B_2$ be two branching processes of net $N$. $B_1$ approximates $B_2$, denoted $B_1 \sqsubseteq B_2$, if there exists an injective homomorphism from $B_1$ to $B_2$.

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## Intuition

$B_1$ approximates $B_2$ whenever every (partial) distributed run in $B_1$ is also contained in $B_2$. In other words, $B_1$ is isomorphic to an initial part of $B_2$. Being an approximation on branching processes is the analogue of being a prefix on sequences.

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## Examples

On the black board. Obviously, $\sqsubseteq$ is a partial order on branching processes.

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## Lemma

Approximation is preserved by isomorphism: if $B_i'$ is isomorphic to $B_i$ (for $i = 1, 2$), then $B_1 \sqsubseteq B_2$ implies $B_1' \sqsubseteq B_2'$. Thus, $\sqsubseteq$ can be extended to a partial order on isomorphism classes (of branching processes).

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## Proof.

Home exercise. Basically juggling with homomorphisms. $\square$

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The set of isomorphism classes of branching processes of net $N$ is a complete lattice with respect to the approximation relation $\sqsubseteq$. Formally, $(\mathbb{B}, \sqsubseteq)$ is a complete partial order, where $\mathbb{B}$ is the set of isomorphism classes of branching processes.

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## Complete lattice

Recall that a complete lattice is a partial order $(\mathbb{B}, \sqsubseteq)$ such that all subsets of $\mathbb{B}$ have LUBs and GLBs.

# The true concurrency semantics of a net

## Corollary: the unfolding of a net

Every one-bounded net has a unique maximal (with respect to $\sqsubseteq$) branching process up to isomorphism. This is called the unfolding or true concurrency semantics of net $N$.

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We denote by $B_{\text{max}} = ((P_{\text{max}}, T_{\text{max}}, F_{\text{max}}), h_{\text{max}})$ a representative of the isomorphism class of the maximal branching process of $N$.

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## Example

On the black board.

# The true concurrency semantics of Petri nets

The true concurrency semantics of a Petri net is given by its unfolding.

Recall: The interleaving semantics of a Petri net is given by its marking graph.

# Overview

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Branching process $B = (P, T, F, M_0)$ is a prefix of $B_{\max}$ if $B \sqsubseteq B_{\max}$ and $P \subseteq P_{\max}$ and $T \subseteq T_{\max}$. $B$ is finite whenever $P$ and $T$ are finite.

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**Finite prefix existence theorem** **[McMillan, 1992]**

For every finite one-bounded net $N$, there exists a finite prefix $B_{\text{fin}}$ of $B_{\max}$ that covers all reachable markings of $N$. The size of the finite prefix can maximally be exponential in the size of $N$.

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## Proof.

Follows directly from two facts:

1. Every reachable marking is represented by some cut of $B_{\max}$, and
2. The set of reachable markings of a finite one-bounded net is finite.

□

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## Fact

For configuration $C$ of $B_{\max}$ (of net $N$), and $x_1 \ldots x_n$ a linearisation of the transitions in $C$ (respecting $\preceq$), the sequence $h_{\max}(x_1) \ldots h_{\max}(t_n)$ is a sequential run of the original net $N$.

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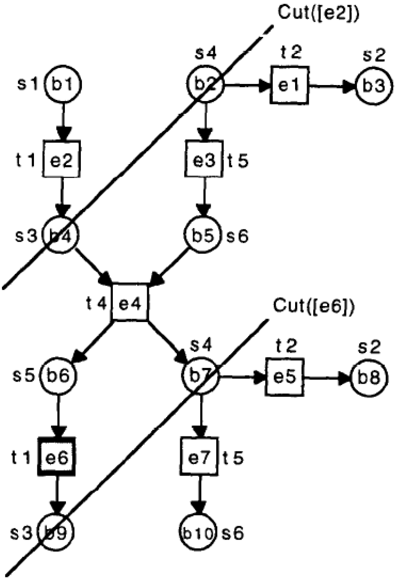
## Intuition

Cuts correspond to markings reached by firing all transitions in a given finite configuration.

# Example


Cut([e2])
s4
t2
s2
s1
b1
b2
e1
b3
t1
e2
e3
t5
s3
b4
b5
s6
t4
e4
Cut([e6])
s4
t2
s2
s5
b6
b7
e5
b8
t1
e6
e7
t5
s3
b9
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s6

# Transition causes

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Let $K = (Q, V, G)$ be an occurrence net and $v \in V$. The set $[v]$ of causes of $v$ is defined by:

$$[v] \;=\; \{\, v' \in V \mid v' \preceq v \,\}.$$

(Recall that $\preceq$ denotes $G^*$, the reflexive and transitive closure of $G$.)

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## Example

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## Facts

1. For each $v$, $[v]$ is a finite configuration.
2. For every configuration $C$ of $K$, either $v \notin C$ or $[v] \subseteq C$.

# Cut-off event

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Let $B_{\max} = ((P_{\max}, T_{\max}, G_{\max}), h_{\max})$. Transition $t \in T_{\max}$ is a cut-off transition if there exists a transition $t' \in T_{\max} \cup \{\perp\}$ such that:

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## Dummy transition

Remark: $\perp$ is a dummy transition having no input places and ${}^\circ B_{\max}$ as output places, for which we let $[\perp] = \varnothing$. This yields that if $M([t]) = M_0$, then $t$ is a cut-off transition.

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## Fact

If $\big|[t']\big| < \big|[t]\big|$ and $M([t]) = M([t'])$, then the "continuations" of $B_{\max}$ from $Cut([t])$ and $Cut([t'])$ are isomorphic.

# McMillan prefix

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## McMillan prefix

The McMillan prefix of one-bounded net $N$ is the branching process $B_{\text{fin}}$, the unique prefix of $B_{\text{max}}$ having $T_{\text{fin}}$ as set of transitions satisfying for each $t \in T_{\text{max}}$::

$$t \in T_{\text{fin}} \quad \text{iff} \quad \text{no transition } t' \prec t \text{ is a cut-off transition.}$$

# Computing the McMillan prefix

## Algorithm

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5. Terminate when no further transitions can be added.

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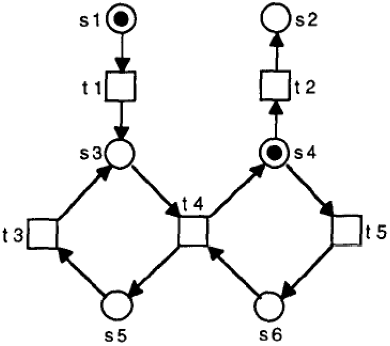
## Remark

Termination is ensured by the finiteness of the number of reachable markings on $N$, as $N$ is one-bounded.

# Example net and one of its branching processes




A sample one-bounded elementary system net

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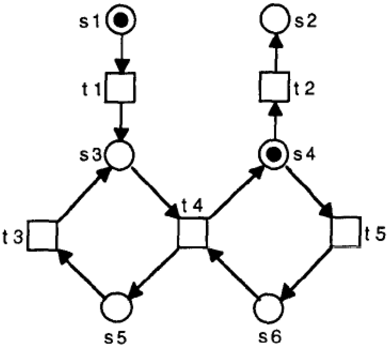


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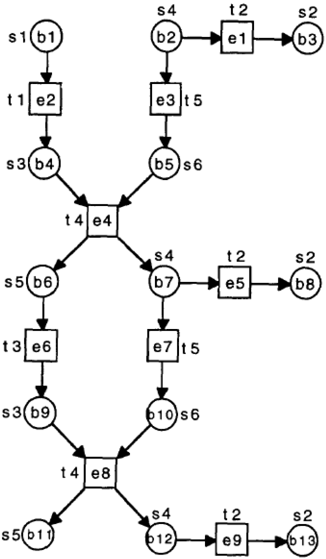
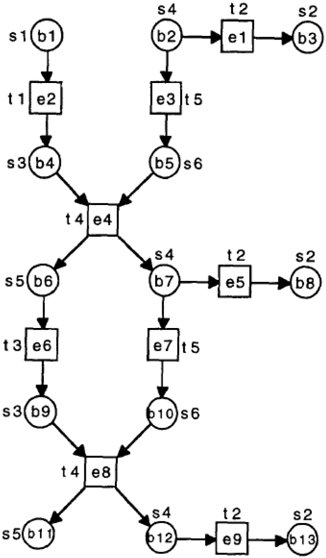


# Its McMillan prefix



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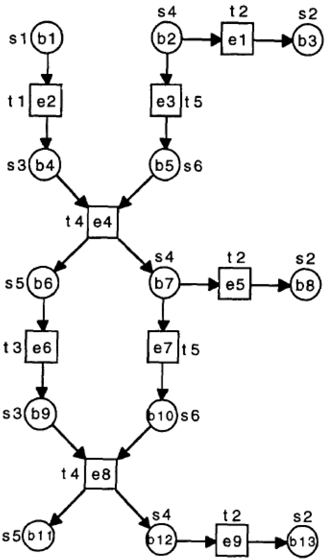
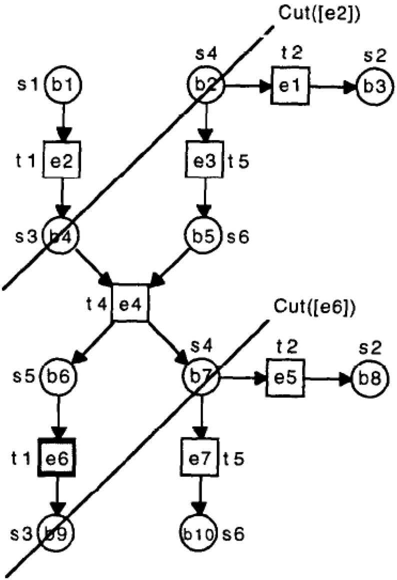
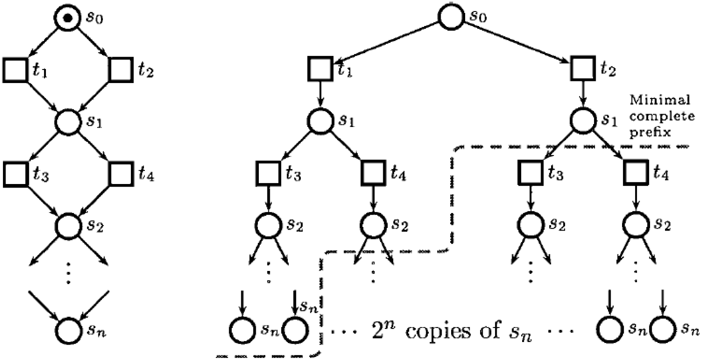


# An exponentially-sized McMillan prefix




For every marking $M$ all the configurations $[t]$ satisfying $M([t]) = M$ have the same size, and therefore there exist no cut-off events [Kishinevsky and Taubin]

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- Isomorphic branching process with $\sqsubseteq$ are a complete lattice
- True concurrency semantics of $N$ = the maximal element (under $\sqsubseteq$)
- For 1-bounded nets, the McMillan prefix covers all reachable markings