

Concurrency Theory

Lecture 17: True Concurrency Semantics of Petri Nets (I)

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<http://moves.rwth-aachen.de/teaching/ws-19-20/ct>

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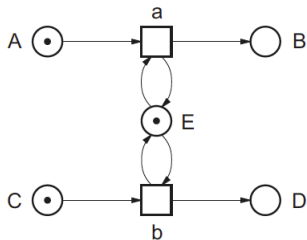
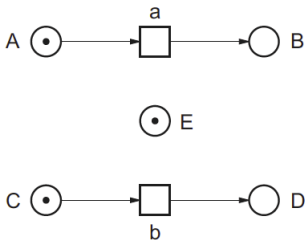
Overview

- 1 Introduction
- 2 Nets and markings
- 3 The true concurrency semantics of Petri nets
- 4 Distributed runs
- 5 Summary

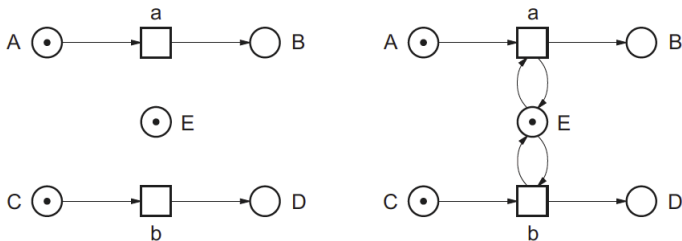
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Motivation

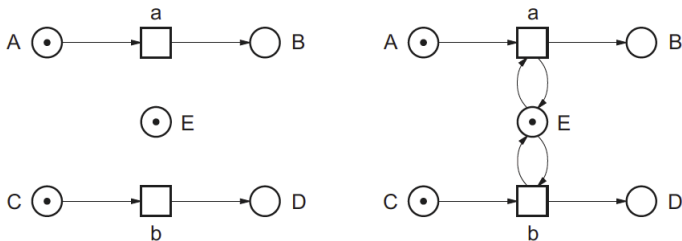


Motivation



Nets with identical sequential runs (a occurs before b , or vice versa), but the left net allows the simultaneous execution of a and b whereas the right one does not.

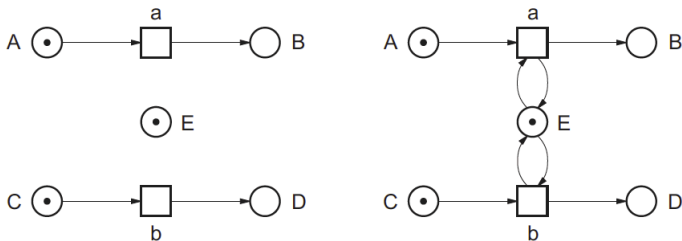
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Interleaving semantics **cannot** distinguish these nets!

This requires a finer perspective on transition execution.

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Nets

Net

A **Petri net** N is a triple (P, T, F) where:

- ▶ P is the countable set of **places**
- ▶ T is the countable set of **transitions** with $P \cap T = \emptyset$
- ▶ $F \subseteq (P \times T) \cup (T \times P)$ are the **arcs**.

Places and transitions are generically called **nodes**.

We assume that $\bullet t$ and t^\bullet are finite, for each $t \in T$.

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Marking

A **marking** M of a net $N = (P, T, F)$ is a mapping $M : P \rightarrow \mathbb{N}$.

For net $N = (P, T, F)$ and marking M_0 , the tuple (P, T, F, M_0) is called an **elementary system net**. M_0 is the **initial marking** of N .

Transition occurrence

Enabling and occurrence of a transition

A marking M **enables** a transition t if $M(p) \geq 1$ for each place $p \in \bullet t$.

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Transition t can **occur** in marking M if t is enabled at M . Its occurrence leads to marking M' , denoted $M \xrightarrow{t} M'$, defined for place $p \in P$ by:

$$M'(p) = M(p) - F(p, t) + F(t, p).$$

where we represent F by its characteristic function.

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$M \xrightarrow{t} M'$ is also called a **step** of the net N .

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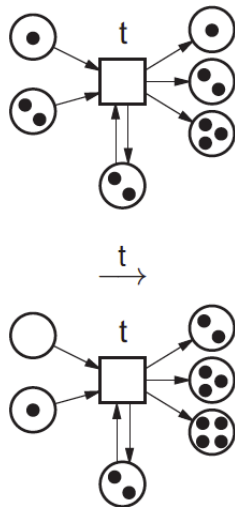
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Reachable markings

Step sequence

A sequence of transitions $\sigma = t_1 t_2 \dots t_n$ is a **step sequence** if there exist markings M_1 through M_n such that:

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots \xrightarrow{t_{n-1}} M_{n-1} \xrightarrow{t_n} M_n$$

Marking M_n is **reached** by the occurrence of σ , denoted $M_0 \xrightarrow{\sigma} M_n$.

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M is a **reachable marking** if there exists a step sequence σ with $M_0 \xrightarrow{\sigma} M$.

Sequential runs

Sequential run

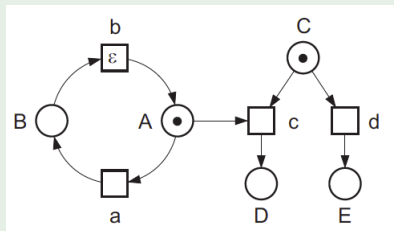
Let N be an elementary net system. A **sequential run** of N is a sequence

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots$$

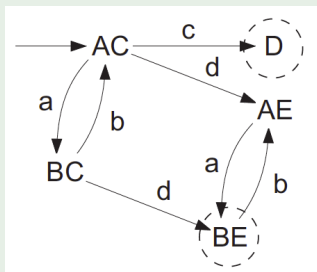
of steps of N starting with the initial marking M_0 . A run can be finite or infinite. A finite run $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots \xrightarrow{t_n} M_n$ is **complete** if M_n does not enable any transition.

Marking graph

The **marking graph** of N has as nodes the reachable markings of N and as edges the reachable steps of N .



A sample elementary net system



Its marking graph

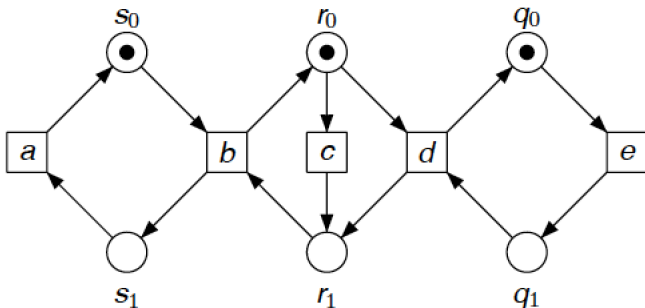
The interleaving semantics of Petri nets

The interleaving semantics of a Petri net is its marking graph.

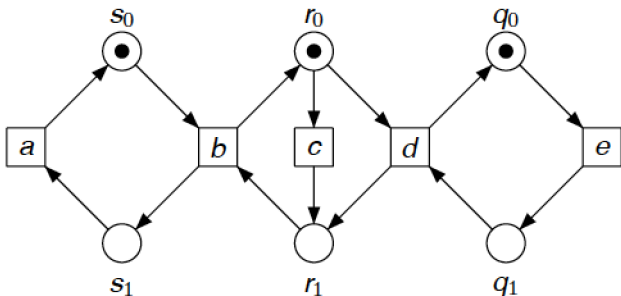
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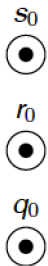
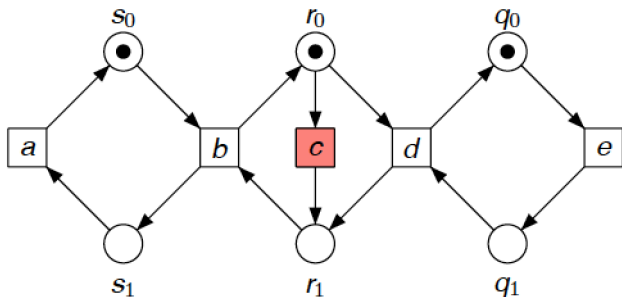
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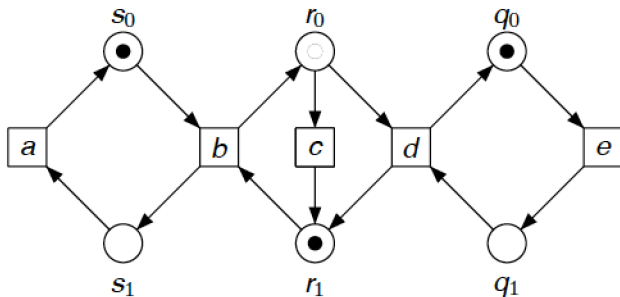
The true concurrency semantics of Petri nets

 s_0  r_0  q_0 

The true concurrency semantics of Petri nets



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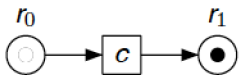
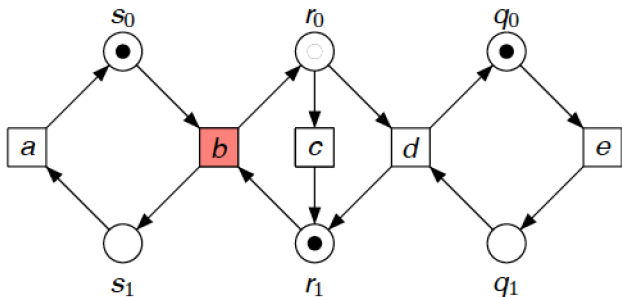

 s_0

 r_0

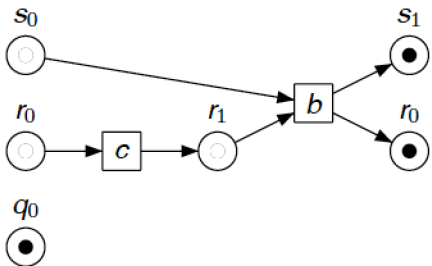
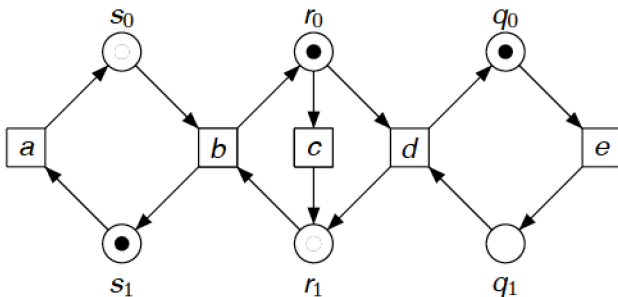
 r_1

 q_0

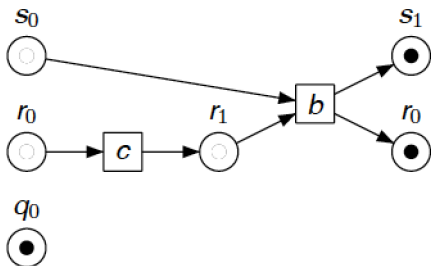
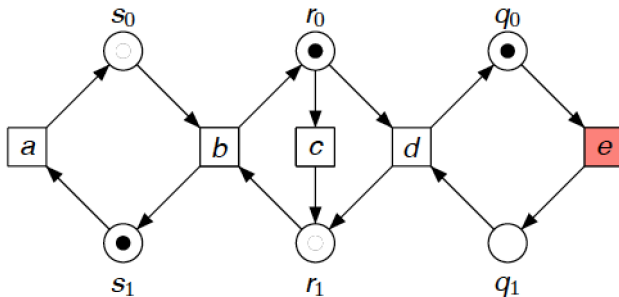

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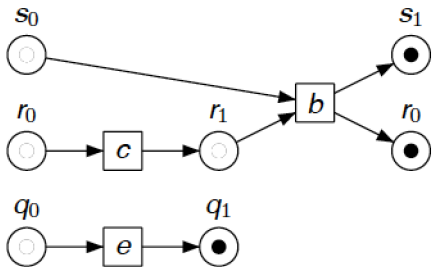
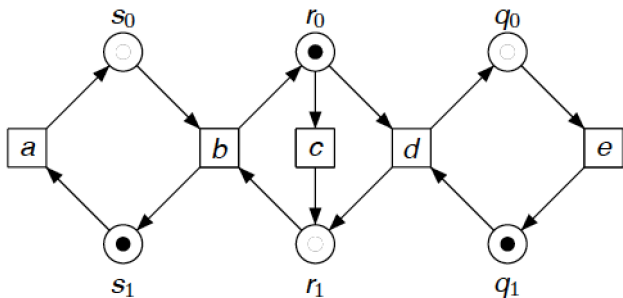
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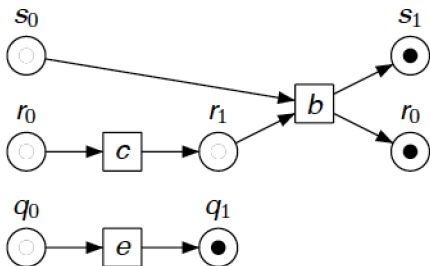
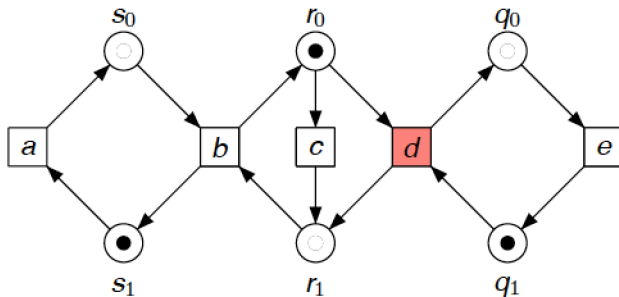
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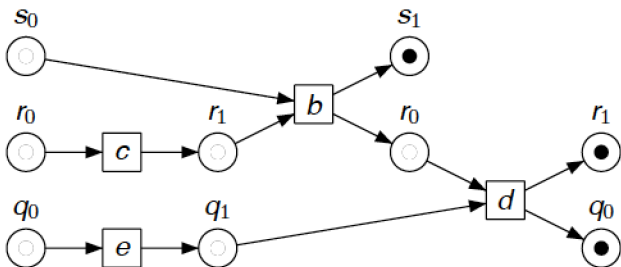
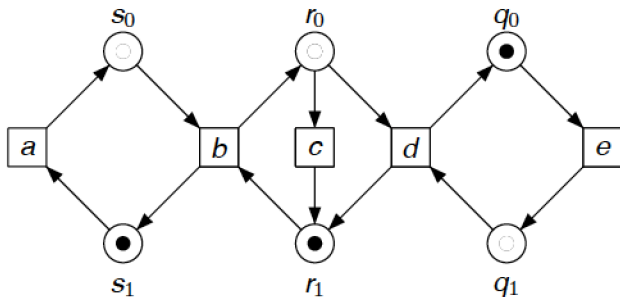
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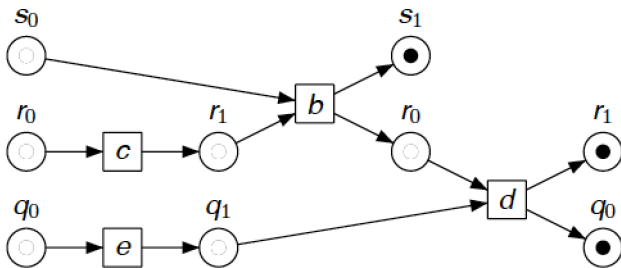
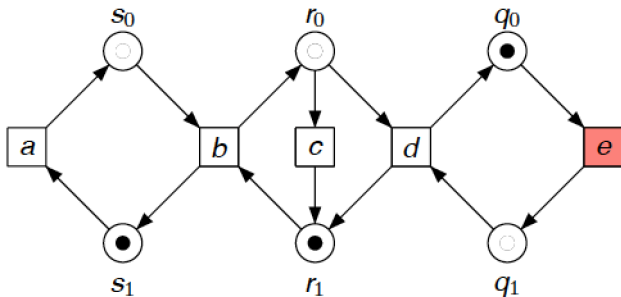
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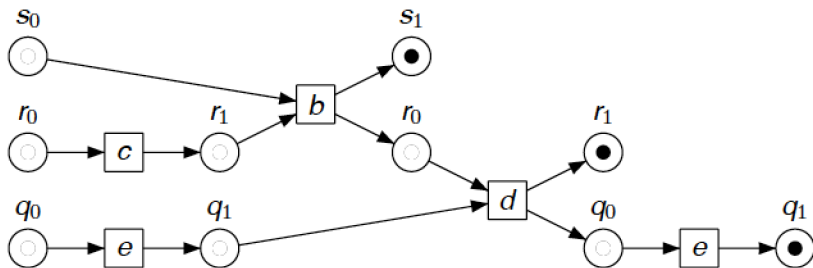
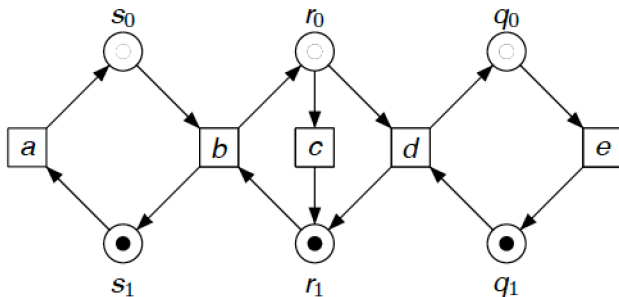
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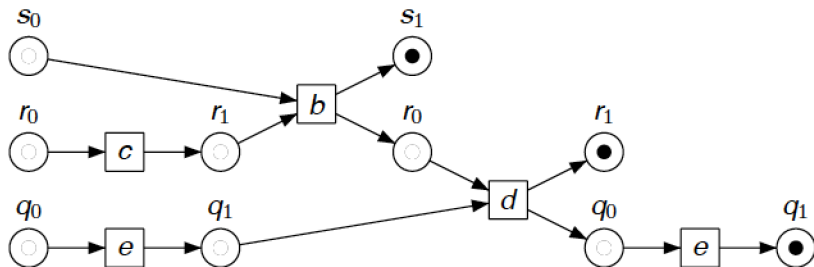
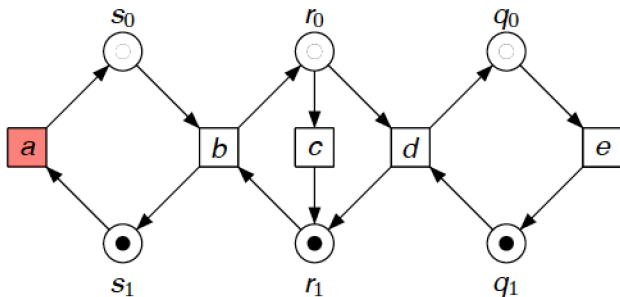
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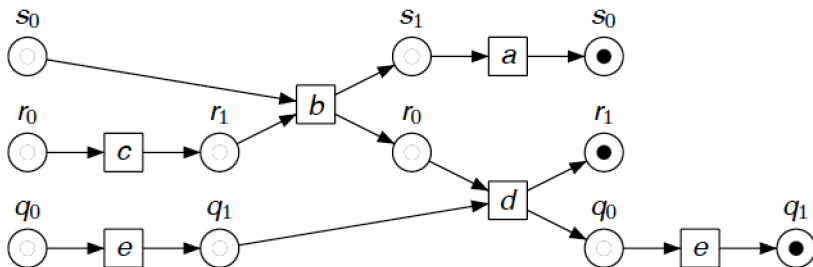
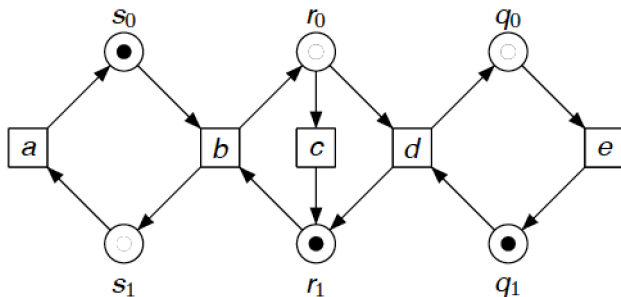
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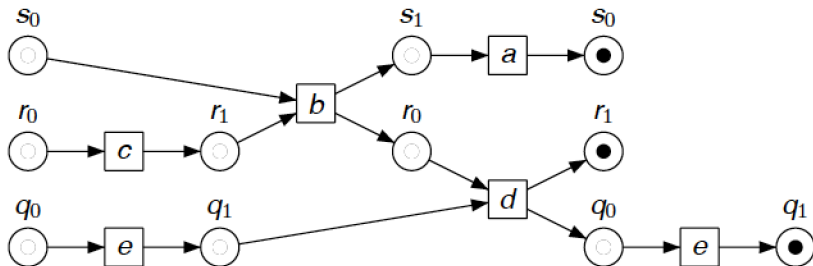
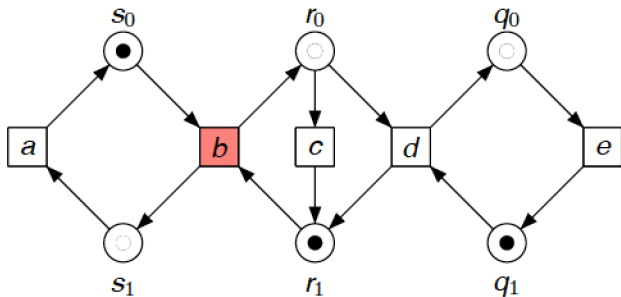
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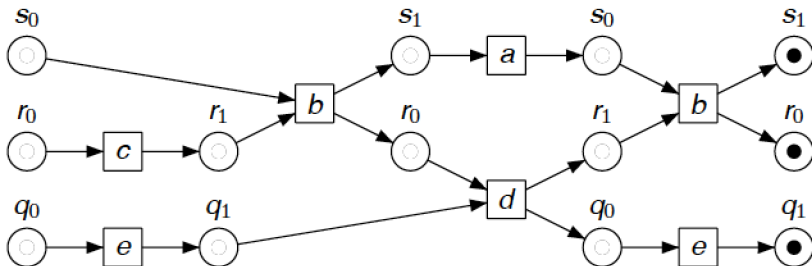
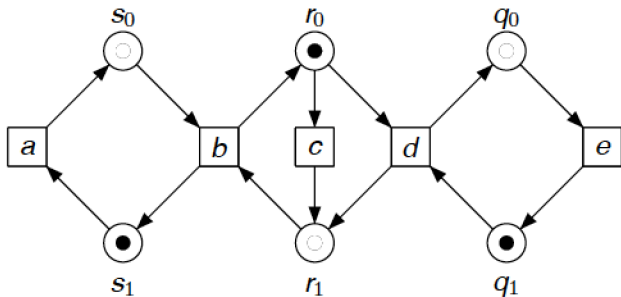
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Interleaving versus true concurrency

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The interleaving thesis:

The total order assumption is a reasonable abstraction, adequate for practical purposes, and leading to nice mathematics

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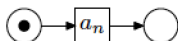
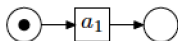
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The true concurrency thesis:

The total order assumption does not correspond to physical reality and leads to awkward representations of simple phenomena

Interleaving versus true concurrency

In interleaving semantics, a system composed of n independent components



has $n!$ different executions

The automaton accepting them has 2^n states

In true concurrency semantics, it has only one nonsequential execution

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Actions

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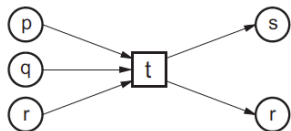
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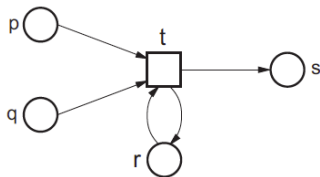
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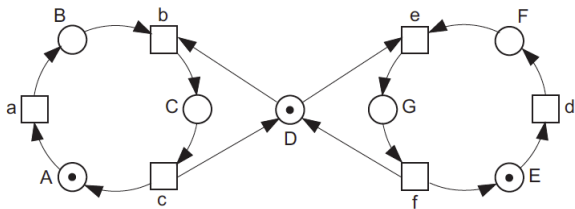


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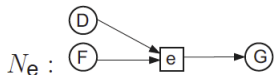
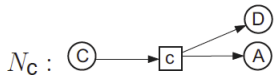
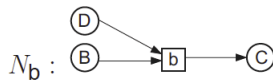
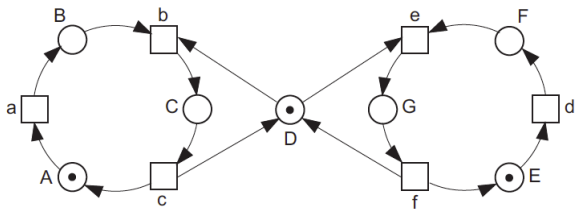


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Mutual exclusion net and its actions



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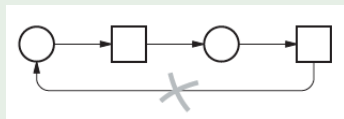
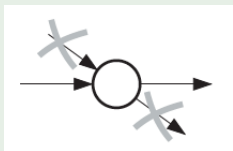
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Intuition

No place branches, no sequence of arcs forms a loop, and each sequence of arcs has a first node.



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3. for each node $x \in Q \cup V$, the set $\{y \mid (y, x) \in G^+\}$ is finite

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3. Each sequence of arcs (flows) has a first element
4. The initial marking contains all places without incoming arcs

Causal net

A (possibly infinite) net $K = (Q, V, G, M_0)$ is called a **causal** net iff:

1. for each $q \in Q$, $|\bullet q| \leq 1$ and $|q \bullet| \leq 1$
2. the transitive closure (called **causal order**) G^+ of G is irreflexive
3. for each node $x \in Q \cup V$, the set $\{y \mid (y, x) \in G^+\}$ is finite
4. M_0 equals the minimal set of places in K under G^+ , i.e.,

$$M_0 = {}^\circ K = \{q \in Q \mid \bullet q = \emptyset\}.$$

Causal nets

A **causal** net constitutes the basis for a distributed run.

It is a possibly infinite net which satisfies:

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Note: the “runs” of the example net (with initial marking) are all causal nets

Properties of causal nets

Properties of causal nets

Lemma

Let $N = (P, T, F, M_0)$ be a causal net. Then every step sequence:

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots \xrightarrow{t_k} M_k$$

of net N satisfies $M_j \cap t_k^\bullet = \emptyset$ for all $j = 0, \dots, k-1$.

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Proof.

By contraposition.

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Boundedness of causal nets

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Every causal net is one-bounded, i.e., in every marking every place will hold at most one token.

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Boundedness of causal nets

Every causal net is one-bounded, i.e., in every marking every place will hold at most one token.

Proof.

Follows directly from the fact that the initial marking M_0 is one-bounded, and by the above lemma. □

Completeness of a causal net

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Absence of superfluous places and transitions

Let $N = (P, T, F, M_0)$ be a causal net. Then there exists a possibly infinite step sequence

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots \xrightarrow{t_k} M_k \xrightarrow{t_{k+1}} \dots$$

of N such that $P = \bigcup_{k \geq 0} M_k$ and $T = \{t_k \mid k > 0\}$.

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Proof.

On the black board. □

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Proof.

On the black board. □

A causal net thus contains no superfluous places and transitions, as every place is visited and every transition is fired in the above step sequence.

Outset and end of a causal net

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Outset and end of a causal net

The **outset** and **end** of causal net $K = (Q, V, G, M)$ are defined by:

$${}^{\circ}K = \{q \in Q \mid \bullet q = \emptyset\} \quad \text{and} \quad K^{\circ} = \{q \in Q \mid q^{\bullet} = \emptyset\}.$$

Places without an incoming arc form the outset ${}^{\circ}K$. The places without an outgoing arc form the end K° .

What is a distributed run?

Distributed run

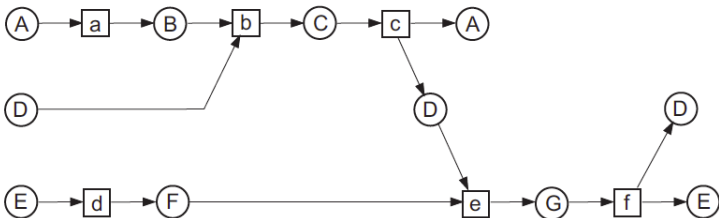
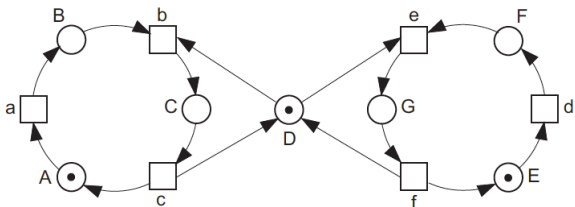
A **distributed run** of a one-bounded elementary net system N is:

1. a **labeled** causal net K_N
2. in which each transition t (with $\bullet t$ and $t\bullet$) is an **action** of N .

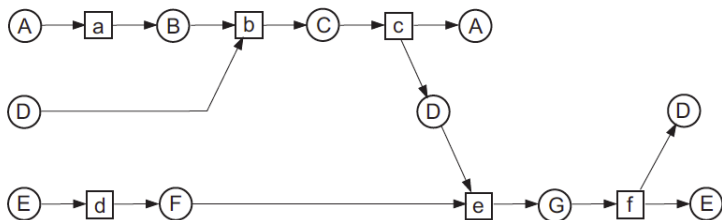
A distributed run K_N of N is **complete** iff (the marking) $\circ K$ represents the initial marking of N and (the marking) K_N° does not enable any transition.

If N is clear from the context we just write K for K_N .

A distributed run for mutual exclusion



A distributed run for mutual exclusion



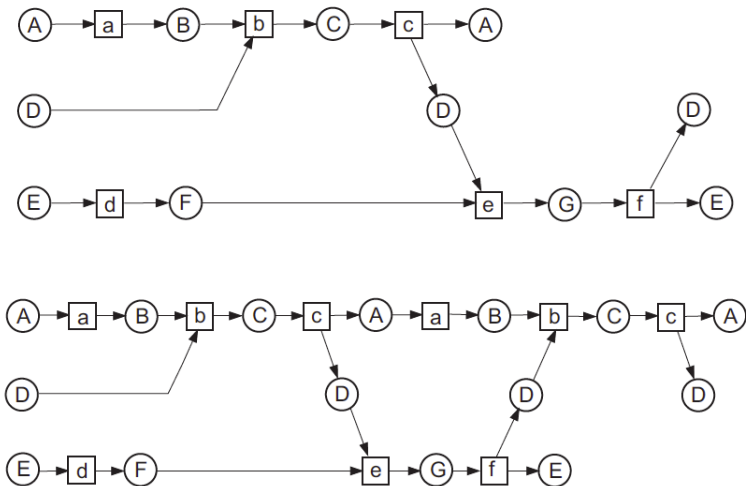
Distributed run of the mutual exclusion algorithm.

Actions N_a , N_b , N_c and N_d **causally precede** N_e . They form a chain.

N_a and N_d are not linked by actions; they are **causally independent**.

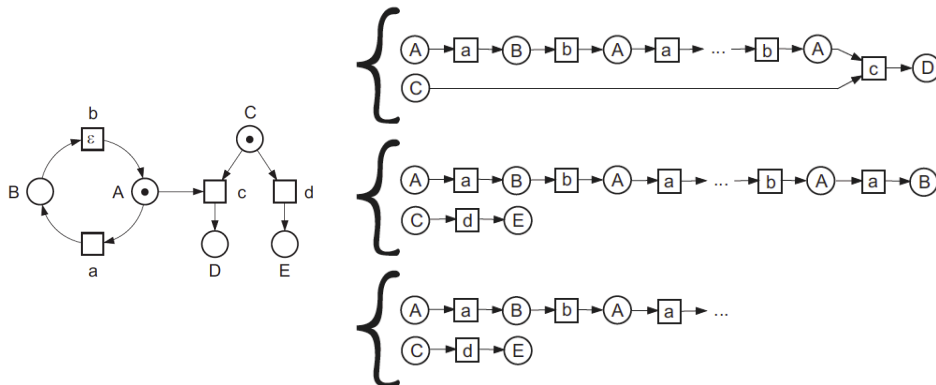
The same applies to N_b and N_d and N_c and N_d .

Expansion of a distributed run for mutual exclusion



A distributed run (top) and its extension with actions b and c (below).

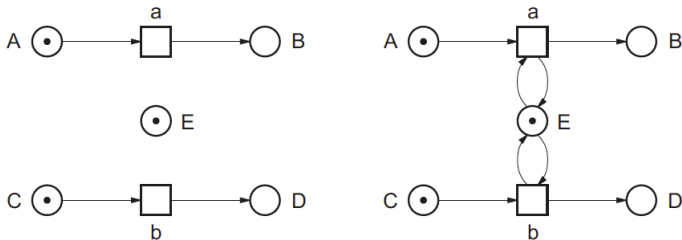
More distributed runs



Various finite distributed runs and an infinite distributed run (right) of net (left).

Causal order

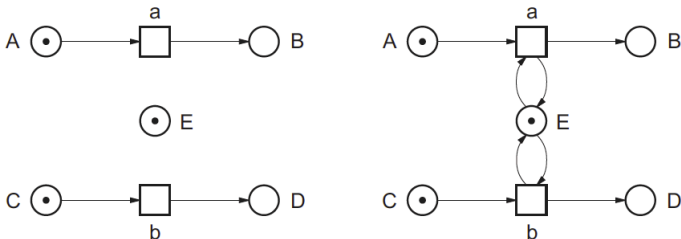
Opposed to sequential runs, distributed runs show the **causal order** of actions.



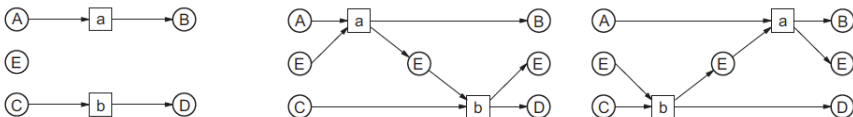
Nets with identical sequential runs (a occurs before b , or vice versa),

Causal order

Opposed to sequential runs, distributed runs show the **causal order** of actions.



Nets with identical sequential runs (a occurs before b , or vice versa), but the left net has the left distributed run below, the right net both other ones:



Composition of distributed runs

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Composition of distributed runs

For $i = 1, 2$, let $K_i = (Q_i, V_i, G_i)$ be causal nets, labeled with ℓ_i . Let $(Q_1 \cup V_1) \cap (Q_2 \cap V_2) = K_1^\circ = {}^\circ K_2$ and for each $p \in K_1^\circ$ let $\ell_1(p) = \ell_2(p)$.

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Intuition

The composition $K \bullet L$ is formed by identifying the end K° of K with the outset ${}^\circ L$ of L . To do this, K° and ${}^\circ L$ must represent the same marking.

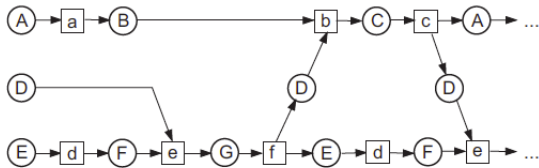
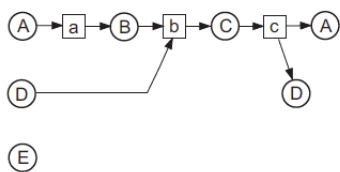
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Examples on the black board.

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Examples on the black board.

Today: a characterization of distributed runs using homomorphisms.

Net homomorphisms

²Here $h(X)$ for set X of nodes is defined by $h(X) = \bigcup_{x \in X} h(x)$.

³Due to the 1-boundedness, a marking M is a subset of the set P of places.

Net homomorphisms

Homomorphism

A **homomorphism** from $N_1 = (P_1, T_1, F_1, M_{0,1})$ to $N_2 = (P_2, T_2, F_2, M_{0,2})$ is a mapping $h : P_1 \cup T_1 \rightarrow P_2 \cup T_2$ such that:

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Intuition

A homomorphism is a mapping between nets that preserves the nature of nodes and the environment of nodes. A homomorphism from N_1 to N_2 means that N_1 can be folded onto a part of N_2 , or in other words, that N_1 can be obtained by partially **unfolding** a part of N_2 .

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Distributed run using homomorphisms

Distributed run

[Best and Fernandez, 1988]

A **distributed run** of an elementary net system N is a pair (K, h) where K is a causal net and h is a homomorphism from K to N .⁴

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A **distributed run** of an elementary net system N is a pair (K, h) where K is a causal net and h is a homomorphism from K to N .⁴

Intuition

A distributed run (K, h) of N may be viewed as a net K of which the places and transitions are labeled by places and transitions of N

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A **distributed run** of an elementary net system N is a pair (K, h) where K is a causal net and h is a homomorphism from K to N .⁴

Intuition

A distributed run (K, h) of N may be viewed as a net K of which the places and transitions are labeled by places and transitions of N such that the labeling h forms a net homomorphism from K to N .⁵

⁴Best and Fernandez called this a process of a net.

⁵In the previous slides, the labeling h was explicitly given as ℓ .

Examples

Overview

- 1 Introduction
- 2 Nets and markings
- 3 The true concurrency semantics of Petri nets
- 4 Distributed runs
- 5 Summary**

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- ▶ Distributed run = the “true concurrency” analogue to a sequential run